Solutions To Assignment 3

1. Let the altitudes of the triangle be AD, BE, and CF. We want to prove that

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1.$$

From $\triangle ACF$, we see that $\angle ACF = 90° - \angle A$ and from $\triangle ABE$, we see that $\angle ABE = 90° - \angle A$. Thus, $\triangle ACF \sim \triangle ABE$, and so $\frac{AF}{AC} = \frac{AE}{AB}$, and this is equivalent to $\frac{AF}{AE} = \frac{AC}{AB}$.

Similarly, $\triangle BDA \sim \triangle BFC$ by the same argument, and so $\frac{BD}{BA} = \frac{BF}{BC}$. This is equivalent to $\frac{BD}{BF} = \frac{BA}{BC} = \frac{AB}{BC}$.

Finally, $\triangle CDA \sim \triangle CEB$, and so $\frac{CD}{CA} = \frac{CE}{CB}$. This is equivalent to $\frac{CE}{CD} = \frac{CB}{CA} = \frac{BC}{AC}$.

Therefore, we have $\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = \frac{AF}{AE} \times \frac{BD}{BF} \times \frac{CE}{CD} = \frac{AC}{AB} \times \frac{AB}{BC} \times \frac{BC}{AC} = 1$.

Hence, by Ceva's Theorem, we have proven that the three altitudes are concurrent.

2. *Solution 1:* Label the parallelogram $ABCD$. Let the diagonals of the quadrilateral meet at M, and let the areas of triangles AMB, BMC, CMD, and DMA be a, b, c, and d, respectively. Since the diagonals divide the quadrilateral into two triangles of equal area, we have $a + b = c + d$ and $a + d = b + c$. Simplifying these two equations gives $b = d$ and $a = c$.

Look at triangles DAM and BAM. Both triangles have the same height, and so the ratio of their areas is the ratio of their bases. In other words, we have $\frac{DM}{MB} = \frac{d}{a} = \frac{b}{a}$. Similarly, if we look at triangles DCM and BCM, we have $\frac{DM}{MB} = \frac{c}{b} = \frac{a}{b}$. Therefore we have proven that $\frac{b}{a} = \frac{a}{b}$, and because a and b are both positive, this implies that $a = b$.

Therefore, $a = b = c = d$, and so all four of these mini-triangles have the same area. But more importantly, $\frac{DM}{MB} = \frac{b}{a} = 1$, since $a = b$, and so this implies that $DM = MB$. By symmetry, we have $AM = MC$. Thus, M is the midpoint of both AC and BD. In other words, the diagonals of the quadrilateral bisect each other.

Since $AM = MC$, $BM = MD$, and $\angle AMB = \angle CMD$, we have $\triangle AMB \cong \triangle CMD$ by the Side-Angle-Side (SAS) congruency test. Hence $\angle ABM = \angle CDM$, and by the Parallel Line Theorem, AB must be parallel to CD. By symmetry, AD is parallel to BC, and so we have proven that opposite sides of the quadrilateral are parallel, i.e., that $ABCD$ is a parallelogram.

Solution 2: This solution is quite clever, and a few of you came up with this one. Drop a perpendicular from A to diagonal BD, let the intersection point be P. Drop a perpendicular from C to diagonal BD, and let the intersection point be Q. Since diagonal BD divides the quadrilateral into two triangles of equal area, triangles ABD and CBD have equal area. Since both have the same base, their heights must be equal, i.e., $AP = CQ$. Now, $\angle AMP = \angle CMQ$ by the Opposite Angle Theorem and $\angle APM = \angle CQM = 90°$, so $\triangle APM \cong \triangle CQM$ by the Side-Angle-Angle (SAA) congruency test.

Since these two triangles are congruent, it follows that $AM = MC$. Similarly, by dropping the perpendiculars from B and D to diagonal AC, the same argument shows that $BM = MD$. Thus, we have shown that the diagonals of the quadrilateral must bisect each other, and then we carry on the proof the same way as we did above. Hence, we conclude that $ABCD$ is a parallelogram.

3. Let $\angle ADP = \angle CDP = \angle ABE = \angle CBE = x$ and $\angle DAM = \angle BAM = \angle BCN = \angle DCN = y$. Thus, $4x + 4y = 360°$, and so $x + y = 90°$. Thus, $\angle APD = 180° - x - y = 90°$, and so $\angle QPS = 90°$ as well. Similarly, $\angle QRS = 90°$.

   Look at $\triangle DQC$. Since $\angle QDC = x$ and $\angle QCD = y$, we conclude that $\angle DQC = 180° - x - y = 90°$. Thus, $\angle PQR = 90°$, and similarly $\angle PSR = 90°$.

   Thus, each angle in quadrilateral $PQRS$ is 90 degrees, and so we conclude that $PQRS$ is a rectangle.

   Since AM is a bisector of $\angle DAB$, let $\angle DAM = \angle BAM = y$. Since $ABCD$ is a parallelogram, $\angle DMA = \angle BAM = y$, since they are alternate angles. Thus, $\triangle ADM$ is isosceles. Using the same reasoning in $\triangle CBN$, we see that it is also isosceles. Thus, $NB = BC = b$, and so $AN = AB - NB = a - b$.

   Thus, $\triangle ADM$ and $\triangle CBN$ are identical isosceles triangles. Also, $AM \parallel NC$, since corresponding angles are equal (i.e., $\angle AMD = \angle NCD = y$. This implies that $AP \parallel NR$.

   Since $AD = BN = b$, triangles APD and NRB are congruent, and so $AP = NR$. Since $AP = NR$ and $AP \parallel NR$, we conclude that $APRN$ must be a parallelogram. Thus, $AN = PR$, and since $AN = a - b$, we have proven that $PR = a - b$.

4. *Solution 1:* Connect MO. By the Hypotenuse-Side congruency test, $\triangle SOM \cong \triangle TOM$, since $SM = MT$ (M is the midpoint of ST and $OS = OT$. Thus, $\angle SMO = \angle TMO$, and since the sum of these two angles is $180°$, it follows that $\angle SMO = 90°$. Thus, $\angle SPO + \angle SMO = 180°$, and so $SMOP$ is a cyclic quadrilateral.

   Join PM. Now we can apply properties of cyclic quadrilaterals. Since angles subtended by the same chord are equal, $\angle SPM = \angle SOM$. But $\triangle SOM \cong \triangle TOM$, and so $\angle SOM = \angle TOM$. Thus, if we let $\angle SPM = x$, then $\angle SOT = 2x$.

   Now $ST = 2$, which is given, and $SO = TO = 2$, since both are radii of a circle with diameter 4. Therefore, $\triangle SOT$ is an equilateral triangle, and so $\angle SOT = 2x = 60°$, and hence it follows that $\angle SPM = 30°$.

   Solution 2: Reflect S about diameter AB to point U. Thus, U is on the circle and $SP = UP$. Furthermore, $SM = MT$, and so $\triangle SPM \sim \triangle SUT$, with $\triangle SPM$ being exactly half that of $\triangle SUT$.

   Thus, if we let $\angle SPM = x$, then $\angle SUT = x$ as well. Let V be the point on the circle that is diametrically opposite S. In other words, pick V so that SV is a diameter of the circle. Thus, $STVU$ is a cyclic quadrilateral, and so $\angle SUT = \angle SVT = x$, because angles subtended by the same chord are equal. But SV is a diameter, and so $\angle STV = 90°$. So STV is a right-angled triangle, with $ST = 2$ and $SV = 4$. So $\sin x = \frac{2}{4} = \frac{1}{2}$, which implies that $x = 30°$. Therefore, $\angle SPM = 30°$.

5. Let $\angle BAD = \angle CAD = x$ and $\angle ABI = \angle CBI = y$. Then $ACDB$ is a cyclic quadrilateral, and because angles subtended by the same chord are equal, we have $\angle BCD = \angle BAD = x$ and $\angle CBD = \angle CAD = x$. Thus, $\angle BCD = \angle CBD$, and so $DB = DC$.

   Now $\angle DBI = x + y$. Notice that $\angle BID = \angle BAI + \angle ABI = x + y$, by the Exterior Angle Theorem. Thus, $\angle DBI = \angle BID$, and so $DB = DI$.

   Therefore, we have proven that $DI = DB = DC$.

6. a) Let $\angle DAE = x$ and $\angle FAE = y$. We wish to prove that AE bisects $\angle DAF$, i.e., that $x = y$. Since AE is the bisector of right angle BAC, we have $\angle BAE = \angle CAE = 45°$. Thus, $\angle BAD = 45° - x$ and $\angle CAF = 45° - y$. Since AD is an altitude of the triangle, $\angle ADB = 90°$, and so $\angle ABD = 45° + x$. Finally, since $\angle ABD = \angle ABC = 45° + x$ and $\angle BAC = 90°$, we have $\angle ACB = \angle ACF = 45° - x$.

   Construct the circumcircle of BAC, and let O be its centre. Since $\angle BAC = 90°$, we have $\angle BOC = 180°$, and so O must lie on BC. In other words, BC is a diameter of the circle. Thus, its midpoint must be the centre of the circle. Since AF is a median, it follows that F is the midpoint of BC, and so F is the centre of the circle. In other words, O and F represent the same point. This means that $FA = FB = FC$. Specifically, $FA = FC$, and so $\angle FAC = \angle FCA$. This works out to $45° - y = 45° - x$, and so $x = y$. Therefore, we have proven that $\angle DAE = \angle FAE$, i.e., that AE bisects $\angle DAF$.

   b) We are given that $AD = 28$ and $AE = 35$. Since $\angle ADE = 90°$, we apply the Pythagorean Theorem to get $DE = 21$. Now by part a), $\angle DAE = \angle FAE$, and so by the Internal Angle Bisector Theorem, we have $\frac{AD}{DE} = \frac{AF}{FE}$. But $AD = 28$ and $DE = 21$, so $\frac{AF}{FE} = \frac{28}{21} = \frac{4}{3}$. So let $AF = 4r$ and $FE = 3r$ for some real number $r > 0$.

   Now $\triangle ADF$ is right-angled at $\angle D$, and so by the Pythagorean Theorem, $AD^2 + DF^2 = AF^2$, i.e., $28^2 + (21 + 3r)^2 = (4r)^2$. If we simplify this quadratic, we get $r^2 - 18r - 175 = (r - 25)(r + 7) = 0$. Since $r > 0$, we must have $r = 25$.

   Thus, $FA = FB = FC = 100$, and so $BC = FB + FC = 200$. Since $AD = 28$, we conclude that the area of $\triangle BAC = \frac{1}{2} \cdot 200 \cdot 28 = 2800$.

7. a) By Ptolemy's Theorem, $PA \cdot BC = PB \cdot AC + PC \cdot AB$. But $\triangle ABC$ is equilateral, and so $BC = AC = AB = k$, for some $k > 0$. Thus, $PA \cdot k = PB \cdot k + PC \cdot k$, and since $k \neq 0$, this simplifies to $PA = PB + PC$, as required.

   b) Each angle of a regular hexagon is $120°$, and so $\angle BCD = \angle GCL = 120°$. Since FC and CJ are bisectors of these two angles, respectively, we have $\angle BCF = \angle GCJ = 60°$. Since F, C, and J are collinear, $\angle BCG = 180° - \angle BCF - \angle GCJ = 60°$.

   Since $BGOC$ is a cyclic quadrilateral, $\angle BOG = \angle BCG = 60°$. Also, $\angle GBO = \angle GCO = \angle GCJ = 60°$. Since $\angle BOG = \angle GBO = 60°$, we have $\angle BGO = 60°$ as well, and so $\triangle BOG$ is equilateral.

Let $CB = a$ and $CG = b$. Let M be the midpoint of FC. Since $\triangle FAM$ and $\triangle MBC$ are both equilateral triangles, we have $FC = FM + MC = AF + BC = a + a = 2a$. So $FC = 2a$, and similarly, $CJ = 2b$. Thus, $FJ = FC + CJ = 2a + 2b$.

By the result from part a), since $\triangle BOG$ is an equilateral triangle, $CB + CO = CG$, so $CO = CG - CB = b - a$.

Thus, $FO = FC + CO = 2a + (b - a) = a + b = \frac{FJ}{2}$. This proves that O is the midpoint of FJ, Q.E.D.

Bonus: Construct the orthocentres of $\triangle ABC$ and $\triangle DEF$. Call the orthocentres H_1 and H_2, respectively. Also construct the centroids of these two triangles, and call them G_1 and G_2, respectively. Since the Euler line connects the circumcentre, centroid, and orthocentre of any triangle, we can draw OG_1H_1 and OG_2H_2. Note that the circumcentre of $\triangle ABC$ is the same point as the circumcentre of $\triangle DEF$, since these six points are concyclic.

We shall show that the midpoint of H_1 and H_2 is *unique*, i.e., regardless of how we pick the triangles, this midpoint will always be the same point. This will prove that Captain Victoria only has to dig at that one point to recover the treasure!

Pick any two other pairs of triangles. By the Pigeonhole Principle, two of the three vertices from $\triangle ABC$ will appear as vertices in the same triangle. For example, if we pick $\triangle DBC$ and $\triangle AEF$, then vertices B and C appear in $\triangle DBC$. Without loss of generality, suppose that we pick $\triangle DBC$ and $\triangle AEF$. We will show that the midpoint of the orthocentres of these two triangles is also T.

Let M be the midpoint of BC. Then in $\triangle ABC$, $AG_1 = 2G_1M$, since the centroid divides the median in a 2 to 1 ratio. Similarly, if G_3 is the centroid of $\triangle DBC$, then $DG_3 = 2G_3M$.

Then $\frac{AG_1}{G_1M} = \frac{DG_3}{G_3M} = 2$, and so G_1G_3 is parallel to AD, by the Side-Splitting Theorem. Furthermore, $G_1G_3 = \frac{AD}{3}$.

If we perform the same construction for $\triangle DEF$ and $\triangle AEF$, we will find two centroids, G_2 and G_4. Letting N be the midpoint of EF, we find that $\frac{AG_4}{G_4N} = \frac{DG_2}{G_2N} = 2$, so G_2G_4 is parallel to AD and $G_2G_4 = \frac{AD}{3}$.

Thus, G_1G_3 is parallel and equal to G_2G_4. This proves that quadrilateral $G_1G_3G_2G_4$ is a parallelogram.

Let U be the midpoint of G_1G_2. Then the midpoint of G_3G_4 is also U, since the diagonals of a parallelogram bisect each other. Thus, no matter how we pick the two triangles from A, B, C, D, E, F, the midpoint of the centroids of these two triangles must always be U.

So regardless of what two triangles we select, U is fixed. From what we know about the Euler line, $\frac{H_1G_1}{G_1O} = \frac{H_2G_2}{G_2O} = 2$, and so G_1G_2 is parallel to H_1H_2 and $H_1H_2 = 3G_1G_2$. Therefore, O, U, and T must be collinear, with $OT = 3OU$, where U is a unique fixed point.

Thus, T must be a unique fixed point as well. Thus, no matter how we pick the two triangles from A, B, C, D, E, or F, the midpoint of the orthocentres of these two triangles must always be the point T.

Therefore, Captain Victoria only needs to dig in one place to recover the treasure.