Solutions To Assignment 4

1. There were n people invited to the party, and $n-1$ came to the reception. Since each pair of people at the reception shook hands, there were a total of $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$ handshakes.

   Say Karin shook p hands. Then $0 < p < n-1$, since Karin only shook hands with *some* of the invited guests, because she came late (as usual). So there were a total of $\frac{(n-1)(n-2)}{2} + p$ handshakes.

   Thus, $\frac{(n-1)(n-2)}{2} + 0 < 73 < \frac{(n-1)(n-2)}{2} + n - 1$, since there were a total of 73 handshakes.

   The inequality $\frac{(n-1)(n-2)}{2} + 0 < 73$ simplifies to $n^2 - 3n < 144$, and from this we can see that n is at most 13, because $14^2 - 3 \cdot 14 = 154 > 144$.

   The inequality $73 < \frac{(n-1)(n-2)}{2} + n - 1$ simplifies to $n^2 - n > 146$, and from this we can see that n is at least 13, because $12^2 - 12 = 132 < 146$.

   Since we require $n \leq 13$ and $n \geq 13$, we conclude that $n = 13$, and so n is unique. Thus, $73 = \frac{(n-1)(n-2)}{2} + p = \frac{(12)(11)}{2} + p = 66 + p$, and so $p = 7$. Hence, we conclude that Karin shook exactly 7 hands.

2. *Solution 1:* Use conditional probability. Let A_p be the event that prisoner A is pardoned, C_p be the event that prisoner C is pardoned, and W_b be the event that the warden says prisoner B will be executed.

   We wish to determine $P(A_p|W_b)$ and $P(C_p|W_b)$, and see if they both equal one-half. Recall from class that the $|$ symbol means "given that".

   From conditional probability, we know that

   $$P(x|y) = \frac{P(x \cap y)}{P(y)}.$$

   Thus, $P(A_p|W_b) = \frac{P(A_p \cap W_b)}{P(W_b)}$ and $P(C_p|W_b) = \frac{P(C_p \cap W_b)}{P(W_b)}$.

   By the scenarios given, it is just as likely for the warden to say that B will be executed as it is for the warden to say that C will be executed. Hence, $W_b = W_c$, and because the sum of these two probabilities is 1, it follows that $W_b = \frac{1}{2}$.

   If C is to be pardoned, the warden will automatically say that B will be executed, and so $P(C_p \cap W_b) = P(C_p) = \frac{1}{3}$.

   However, if A is to be pardoned, the warden will say either B or C, depending on the flip of a coin. Hence, $P(A_p \cap W_b) = P(A_p) \cdot \frac{1}{2} = \frac{1}{6}$.

   Therefore, $P(A_p|W_b) = \frac{P(A_p \cap W_b)}{P(W_b)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$, and $P(C_p|W_b) = \frac{P(C_p \cap W_b)}{P(W_b)} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$.

   Hence, we conclude that if the warden revealed that prisoner B would be executed, then A still has a one-third chance of being pardoned, but C chances of surviving has now risen to two-thirds. Thus, the two men did not reason correctly.

Solution 2: Follow the same set-up as above. Since either prisoner A or C got pardoned, it follows that $P(A_p|W_b) + P(C_p|W_b) = 1$.

From above, $P(A_p \cap W_b) = \frac{1}{6}$ and $P(C_p \cap W_b) = \frac{1}{3}$. Thus, if the warden reveals that prisoner B will be executed, C is *twice* as likely as A to be the pardoned prisoner. Thus, $P(C_p|W_b) = 2P(A_p|W_b)$, and since $P(C_p|W_b) + P(A_p|W_b) = 1$, it follows that $P(C_p|W_b) = \frac{2}{3}$ and $P(A_p|W_b) = \frac{1}{3}$.

Solution 3: Let the men A, B, and C be equivalent to doors A, B, and C of the Monty Hall problem. We will let execution be the goats and the pardon be the car (obviously preferred unless they happen to be enrolled in Math 2790). In effect, the prisoner "chooses" door A, and the warden reveals another "door" (person) that will be executed (B). From what we know in the Monty Hall problem, the probability that A is pardoned is $\frac{1}{3}$ (staying), and the probability that C is pardoned is $\frac{2}{3}$ (switching). Thus prisoner A still has the same chance of getting the pardon, but prisoner C's chances of surviving have now doubled. Thus, the two men did not reason correctly.