Solution To Big Wheel Problem

We will assume that each of the twenty numbers occurs with equal probability (a safe assumption), so the probability of getting a given number on the first spin is $\frac{1}{20}$.

Now, we'll list all the possible ways Jacob can get each of the following scores, if he spins the wheel *twice*. For example, he can get 20 exactly three ways: $(15, 5)$, $(10, 10)$, and $(5, 15)$.

Score	5	10	15	20	25	30	35	40	45	50
Ways	0	1	2	3	4	5	6	7	8	9

Score	55	60	65	70	75	80	85	90	95	100	OVER
Ways	10	10	10	10	10	10	10	10	10	10	55

If Jacob does not get over 50 on his first spin (which occurs with probability one-half), he will spin again, and there are 55 different ways he can get over on his second spin. So the probability of Jacob going over is $\frac{1}{2} \cdot \frac{55}{200} = \frac{55}{400}$. If Jacob goes over, then Ryan automatically wins the game.

Consider the number of ways Jacob can get $5i$ points, for $i = 1, 2, 3, \ldots, 10$. For each of these ten cases, we'll determine the probability that Ryan wins, *given that* Jacob gets a total of $5i$ points.

For each $i < 10$, there are $i - 1$ ways of scoring $5i$ points with two spins (note that Jacob will not stay if he gets $5i$ points on his first spin, so he will always spin the wheel twice if he ends up with $5i$ points). And there is a one-half chance the Jacob will spin the wheel twice. So the probability of getting $5i$ points ($i \leq 10$) is $\frac{1}{2} \cdot \frac{i-1}{200} = \frac{i-1}{400}$. If Ryan gets above $5i$ on his first spin, then he will win, and if he gets $5i$ or less, he will spin again.

The probability that Ryan will get above $5i$ on his first spin is $\frac{20-i}{20}$. The probability that Ryan will get $5i$ or less is $\frac{i}{20}$. Now, if he gets $5i$ or less, he will spin again. In order to win, he must get a total between $5i + 5$ and 100, and there are $20 - i$ of these numbers. So he must spin one of the $20 - i$ numbers on his second spin so that his total gets into the desired range.

Therefore, the probability that Ryan wins the game, *given that* Jacob spins $5i$, $i \leq 10$ is

$$P(R_w | J_{5i}) = \frac{20 - i}{20} + \frac{i}{20} \cdot \frac{20 - i}{20} = \frac{20 - i}{20} \cdot \frac{20 + i}{20} = \frac{400 - i^2}{400}.$$

Hence, the probability that Ryan wins the game *and* Jacob spins $5i$ is

$$P(R_w \cap J_{5i}) = P(R_w | J_{5i}) \cdot P(J_{5i}) = \frac{400 - i^2}{400} \cdot \frac{i - 1}{400} = \frac{(400 - i^2)(i - 1)}{160000}.$$

Now we consider the case where Jacob gets $5i$ points, for $i = 11, 12, 13, \ldots, 20$. For each of these ten cases, we'll determine the probability that Ryan wins, *given that* Jacob gets a total of $5i$ points.

For each i, there are 10 ways of scoring exactly $5i$ points by spinning twice, and so the probability of scoring $5i$ points with two spins is $\frac{1}{2} \cdot \frac{10}{200} = \frac{1}{40}$. Once again, we must multiply by $\frac{1}{2}$, since there is a fifty percent chance that Ryan will need to spin the wheel a second time (since he got 50 or less on his first spin). Ryan also has a one in twenty chance of scoring exactly $5i$ points on his first spin. So the probability of getting $5i$ points ($11 \leq i \leq 20$) is $\frac{1}{40} + \frac{1}{20} = \frac{3}{40}$. If Ryan gets above $5i$ on his first spin, then he will win. If he gets exactly $5i$, the two will tie and play again, and if he gets less than $5i$, Ryan will have to spin the wheel a second time.

The probability that Ryan will get above $5i$ on his first spin is $\frac{20-i}{20}$. The probability that Ryan will get less than $5i$ is $\frac{i-1}{20}$. In order for Ryan to win, he must get a total between $5i+5$ and 100, and there are $20-i$ of these numbers. So he must spin one of the $20-i$ numbers on his second spin so that his total gets into the desired range.

Therefore, the probability that Ryan wins the game, *given that* Jacob spins $5i$, $11 \leq i \leq 20$ is

$$P(R_w|J_{5i}) = \frac{20-i}{20} + \frac{i-1}{20} \cdot \frac{20-i}{20} = \frac{20-i}{20} \cdot \frac{19+i}{20}.$$

Hence, the probability that Ryan wins the game *and* Jacob spins $5i$ is

$$P(R_w \cap J_{5i}) = P(R_w|J_{5i}) \cdot P(J_{5i}) = \frac{20-i}{20} \cdot \frac{19+i}{20} \cdot \frac{3}{40} = \frac{3(20-i)(19+i)}{16000}.$$

Therefore, the probability that Ryan wins the game after both of them have spun (ignoring ties) is

$$P_1 = \frac{55}{400} + \sum_{i=1}^{10} \frac{(400-i^2)(i-1)}{160000} + \sum_{i=11}^{20} \frac{3(20-i)(19+i)}{16000} = \frac{4073}{8000}$$

Let T be the probability that Ryan and Jacob end up with the same score. If Jacob spins $5i$ (with $i \leq 10$), then Ryan can spin $5i$ in $i-1$ different ways. Both of them will have spun the wheel twice. So the probability that both Ryan and Jacob get $5i$ (with $i \leq 10$) is $\frac{i-1}{400} \cdot \frac{i-1}{400} = \frac{(i-1)^2}{160000}$.

If Jacob spins $5i$ (with $11 \leq i \leq 20$), then Ryan can either spin $5i$ on his first spin (in which case he will stay and force a tiebreaker) or have two spins of x and $5i-x$ to tie Jacob, with $x < 5i$. The probability of Ryan getting $5i$ in this case is $\frac{1}{20} + \frac{i-1}{20} \cdot \frac{1}{20} = \frac{i+19}{400}$. Thus, the probability that both Ryan and Jacob get $5i$ (with $11 \leq i \leq 20$) is $\frac{3}{40} \cdot \frac{i+19}{400} = \frac{3(i+19)}{16000}$.

Therefore, we have:

$$T = \sum_{i=1}^{10} \frac{(i-1)^2}{160000} + \sum_{i=11}^{20} \frac{3(i+19)}{16000} = \frac{2127}{32000}.$$

Hence, Ryan can win the game in any of the following ways: winning immediately, tying then winning, tying twice then winning, tying three times then winning, and so on. This gets us a nice infinite geometric series, which we evaluate using our formula for the sum of an infinite geometric series:

$$
\begin{aligned}
P &= P_1 + TP_1 + T^2P_1 + T^3P_1 + \ldots \\
&= P_1(1 + T + T^2 + T^3 + \ldots) \\
&= P_1 \cdot \frac{1}{1-T} \\
&= \frac{P_1}{1-T} \\
&= \frac{\frac{4073}{8000}}{1 - \frac{2127}{32000}} \\
&= \frac{16292}{29873}
\end{aligned}
$$

Therefore, the probability that Ryan advances to the showcase is $\frac{16292}{29873}$, which is approximately 54.5 percent.