Tour One - Parity

Let's go to McDonalds, since I'm starving, and I'm sure you are too.

"May I take your order?", says the kind cashier.

I say, "I'll have nine Chicken McNuggets and a Sardine Milkshake, which I hear is a popular drink among the Haligonians here."

"Oh you haven't heard?"

"Heard what?"

"We don't sell McNuggets in packages of nine anymore. There was a big labor dispute, and so we are protesting by not selling certain items on our menu. I'm really sorry, but we only sell McNuggets in packages of six and twenty now".

I'm a little annoyed to hear this, because a six-pack is just too small for me, and twenty is a bit too much. However, I purchase twelve McNuggets (two packages of six), and have a satisfying meal.

I wonder what number of McNuggets I could have bought. For example, there's no way I could have bought ten McNuggets, because no combination of six and twenty would give me exactly ten McNuggets. Likewise, I couldn't have gotten eleven, thirteen, or fourteen McNuggets either. But I could have gotten eighteen McNuggets (three packages of six) or twenty-six McNuggets (one package of six and one package of twenty).

Since this is the year 2001, let me ask, could I have bought exactly 2001 McNuggets? Just pretend I was really really hungry, and felt like I would actually want to consume 2001 McNuggets.

Problem 1.1: *Due to a labour dispute at McDonald's, employees have decided to serve Chicken McNuggets only in packages of* 6 *and* 20*. Under this new system, is it possible to purchase exactly* 2001 *Chicken McNuggets?*

No matter what I seem to do, I can't see a way to purchase exactly 2001 McNuggets. Do you?

Why don't we make a list of numbers that we can make, and see if we can notice a pattern. We have already shown that we can purchase 6, 12, 18, 20, 26, 2000, and 2002 McNuggets. I wonder if there is anything that all of these numbers have in common. What do you notice?

Yes, they are all *even* numbers! Find some more numbers that you can make - once again, you see how these numbers are all even! I wonder if that is always the case, and if so, why?

If we wanted to use algebra, we could say we bought x packages of six and y packages of twenty. So we have $6x + 20y$ total McNuggets. Since $6x$ and $20y$ are both even, its sum must also be even, and so we have just shown that *it is only possible to purchase an even number of McNuggets.*

Let's return to our problem. Can we purchase 2001 McNuggets? Well, 2001 is an *odd* number, and so by our analysis above, our answer must be NO!

One very careful point needs to be stressed here. We have now *proven* that it is only possible to purchase an even number of McNuggets. However, we have *not* proven that it is possible to buy *any* even number of McNuggets. Read those two sentences again and make sure you understand the subtle difference between them. In fact, the latter sentence is not even true, because it is impossible to purchase exactly 2 McNuggets! In fact, there are a whole bunch of even numbers that cannot be purchased, such as 2, 4, 8, 10, 14, 16, and 22.

This problem illustrates the idea of *parity*, which is one of the most powerful concepts in problem-solving. Parity simply refers to the evenness or oddness of an integer. For example, we say that 6 and 20 have the same parity, because they are both even. In our problem here, we argued that if it is possible to purchase N McNuggets, then the parity of N must be even. Since 2001 has odd parity, it is impossible to purchase 2001 McNuggets.

Let's solve some more problems, and investigate this idea of parity.

Problem 1.2: *There are* 5 *red marbles and* 6 *green marbles in a jar. Elizabeth plays a strange game. She removes two marbles at a time, with the following rules:*
a) If the marbles are both green, she puts one green marble back.
b) If there is one marble of each colour, she puts one red marble back.
c) If the marbles are both red, she puts one green marble back.
At the end, there will be one marble left. Which colour is it?

(Note: suppose that there are a whole bunch of green marbles lying around, so that if Elizabeth removes two red marbles on her first move, then she can put one of those green marbles back in the jar.)

Play this game and see what you end up with. You should finish with a red marble. Turns out, this is always true. Let us investigate why.

Note that in each turn, the number of marbles in the jar (of both colours) decreases by exactly one. This is because we are always taking away two marbles and putting back one. So eventually we will only have one marble left. Our task is to show that this final marble is red.

In case a) and b), the jar loses one green marble, and in case c), the jar loses two red marbles and gains one green marble. Thus, the number of red marbles in the jar must decrease by 0 or 2 on each turn. So if we have 5 red marbles, we could get 3 red marbles, but we can't ever get 4. In other words, the *parity* of red marbles must always be *odd*, because we start with an odd number of red marbles, and reducing this quantity by two does not change its parity. To use a fancy mathematical term, the parity of the red marbles is *invariant*, i.e., it never changes.

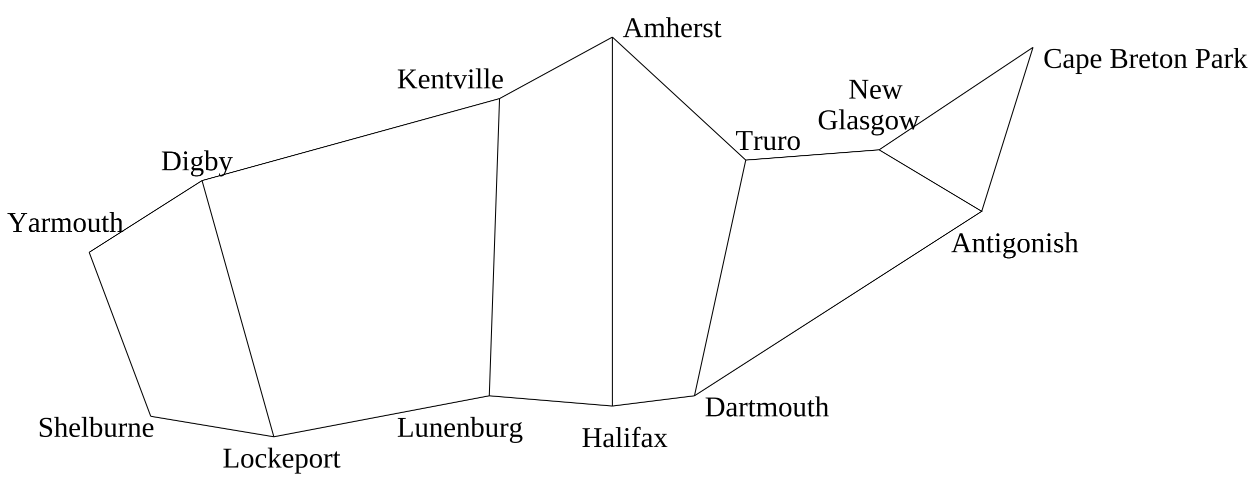
So at the end of the procedure, Elizabeth will either remove all of the red marbles or remove all of the green marbles (and have one marble left of the other colour). But the number of red marbles can never go down to 0, and so Elizabeth cannot ever remove all of the red marbles. So eventually the green marbles will all disappear, and Elizabeth's final marble will be red.

Problem 1.3: *Consider an* 8 *by* 8 *board. I cut two diagonally opposite squares and throw them away. So there are now* 62 *squares left. Can I tile this modified board with exactly* 31 *dominoes? (In other words, can I place the* 31 *dominoes on the board so that every square* *is covered?)*

Colour the squares of the board black and white, like in a checkerboard. Then we have 32 black squares and 32 white squares. When we remove two diagonally opposite squares, we will be removing two squares of the *same* colour. So our 62 squares will consist of 32 black squares and 30 white squares (or vice-versa, it really doesn't matter). Since each domino covers exactly one square of each colour, if we place 31 dominoes on the board, we must cover exactly 31 squares of each colour. However, we have 32 black squares, so the board cannot be completely tiled, and there will be at least one white square that is overlapped. Thus, such a tiling is not possible.

Notice how this idea of comparing black squares to white squares is essentially the same idea as comparing even numbers to odd numbers? Colouring squares is an example of solving a problem using parity. Whenever we can compare two different "states", such as even vs. odd integers, or black vs. white squares, we might be able to solve the problem using parity.

Problem 1.4: *Garrett has stolen a million dollars from Victoria, and has run away with it. Victoria discovers this and wants to get the money back. Victoria starts in Lockeport, and Garrett starts in Yarmouth. Because of road construction, it is only possible to travel between certain cities, as indicated in the diagram below. They take turns moving, and Victoria gets her money back by moving to the city that Garrett is at. Victoria moves first. Can Victoria get her million dollars back? If so, how? If not, explain why.*



This is a wonderful question, and play this game with a friend before reading any further. You play Victoria, so you are trying to catch your friend.

If you played as Victoria, chances are that you were able to catch Garrett after a long sequence of moves, which involved both of you traveling across the entire map. In fact, you almost definitely forced Garrett to move to the very right of the map, and that is how you trapped him.

However, Garrett does not need to be forced anywhere. What if he just keeps moving back and forth, between Yarmouth and Shelburne? Remember, Victoria has to catch Garrett (Garrett can't just turn himself in). If Garrett moves in this manner, how would Victoria ever be able to move to the city Garrett is in, because Garrett will always be one move ahead?

Turns out, Victoria will always be able to catch Garrett, regardless of what Garrett does. In fact, Victoria can always catch Garrett in at most eleven moves, so if she plays correctly, the game is over extremely fast! The key to the problem is Cape Breton. She must go through Cape Breton.

To see why Victoria needs to go through Cape Breton, let's temporarily remove Cape Breton from the map, and also remove all roads going to that city. We can use the same parity argument we used in the previous problem. Colour the following cities white: Yarmouth, Lockeport, Kentville, Halifax, Truro, and Antigonish. Colour every other city black. Then we see that our map has the property that black cities only go to white cities, and vice-versa (the fancy name for a graph with this property is a *bipartite* graph). Since Yarmouth and Lockeport are both white cities, when Victoria moves to start the game, she will be moving to a black city. Now Garrett starts in a white city, so he will also move to a black city. So now it's Victoria's turn to move, and both of them are now on cities of the same colour. This process will repeat forever, because every time it is Victoria's turn to move, the two are on cities of the same colour, and so when she moves, she must be moving to a city of the opposite colour. So any move, Victoria cannot move to the city that Garrett is at, because Garrett's city is always of the opposite colour. So Victoria can never catch Garrett.

Now let's put Cape Breton back on the map. Then our colouring argument no longer works, because Cape Breton is connected to a white city and a black city. This is because if we colour Cape Breton either black or white, then it will be connected to a city with the same colour, and hence, the graph is no longer bipartite. So that is the key to the problem, and if Victoria is to catch Garrett, she must pass through Cape Breton to reverse the parity (so after she passes through Cape Breton Victoria and Garrett are on opposite coloured cities whenever it is Victoria's turn to move). So Victoria can indeed catch Garrett. The quickest way is to make her first six moves be Lockeport to Lunenburg to Halifax to Dartmouth to Antigonish to Cape Breton to New Glasgow. Now Victoria can easily force Garrett into a corner and be able to catch him.

Problem 1.5: *If a, b, and c are odd integers, prove that the polynomial $ax^2 + bx + c$ cannot have a rational root.*

The polynomial ax^2+bx+c has two roots, m and n. We know that $mn = \frac{c}{a}$, since mn is just the product of the roots. Since a and c are integers, $\frac{c}{a}$ is a rational number. Thus, if m is irrational, then so is n, and likewise, if m is rational, then n must be rational as well. We can't have one of m or n being rational and the other being irrational, since their product will be irrational. So let us suppose that ax^2+bx+c has a rational root. Then it must have two rational roots.

If $ax^2 + bx + c$ has rational roots, then it can be factored as the product of two linear terms. So $ax^2 + bx + c = (px + q)(rx + s)$, where p, q, r, and s are integers. Expanding the right side, we have $(px+q)(rx+s) = prx^2 + (ps+qr)x + qs$. Matching coefficients, we have $a = pr$, $b = ps + qr$, and $c = qs$. Since $a = pr$ is odd, both p and r must be odd. Since $c = qs$ is odd, both q and s must be odd. Thus, all four integers, p, q, r, and s are odd. So $ps + qr = b$ must be even, since it is the sum of two odd numbers. However, we are given that b is odd, and we have arrived at a contradiction. Therefore, $ax^2 + bx + c$ cannot have a rational root if a, b, and c are all odd integers.

Here is another way to solve this question. Let $x = \frac{p}{q}$ be a rational root of $ax^2 + bx + c$. So p and q are integers which have no common factor (i.e., gcd(p,q) = 1). Since $\frac{p}{q}$ is a root, by definition we have $a(\frac{p}{q})^2 + b(\frac{p}{q}) + c = 0$, which simplifies to $ap^2 + bpq + cq^2 = 0$, once we multiply both sides of the equation by q^2.

Now, a, b, and c are all odd, as that is what we are given. If p and q are both odd, then each of ap^2, bpq, and cq^2 are odd, and so their sum cannot possibly be 0, an even number.

If p is odd and q is even, then ap^2 is odd, bpq is even, and cq^2 is even. Thus their sum is an odd number, which cannot possibly be 0. Likewise, if p is even, and q is odd, we arrive at a contradiction.

Finally, we cannot have p and q both being even, because we specified that p and q are relatively prime integers (i.e., integers with gcd(p,q) = 1). If p and q are both even, then they both have a factor of 2. So we do not need to consider this case.

Hence, in all cases we have a contradiction, and so there cannot be a rational root of $ax^2 + bx + c$, if a, b, and c are all odd integers.

Here are some more problems to try.

Problem 1.6: There are 7 coins on a table, all tails up. You can flip over exactly two at a time. Can you change them all to heads up? Explain your answer.

Problem 1.7: Twenty-five jealous men live in the unit squares of a 5 by 5 grid. Each of them thinks that his neighbours in adjacent squares (horizontally and vertically) all live better than he does. Is it possible for all of them to move in such a way that everyone ends up in the square of one of his former neighbours?

Problem 1.8: If you've ever played Tetris, you will know that there are five different configurations: a piece can be L-shaped, I-shaped, O-shaped, Z-shaped, or T-shaped. Using each of these pieces exactly once, can you tile a 4 by 5 board? Explain your answer.

Problem 1.9: There are four jars in a room, labeled A, B, C, and D. Jar A contains 9 marbles, jar B contains 10 marbles, jar C contains 13 marbles, and jar D contains 16 marbles. Aaron plays the following game. He selects any three of the jars, draws one marble from each, and puts all three marbles into the fourth jar. He keeps repeating this. Can Aaron play in such a way so that eventually there will be exactly 12 marbles in each jar?

Problem 1.10: Let $f(x) = ax^2 + bx + c$, where a, b, and c are integers. If $f(0) = 15$ and $f(1) = 25$, prove that $f(x)$ cannot have an integral root. Generalize this problem: prove that if $f(x)$ is *any* polynomial, then $f(x)$ cannot have an integral root if $f(0) = 15$ and $f(1) = 25$.