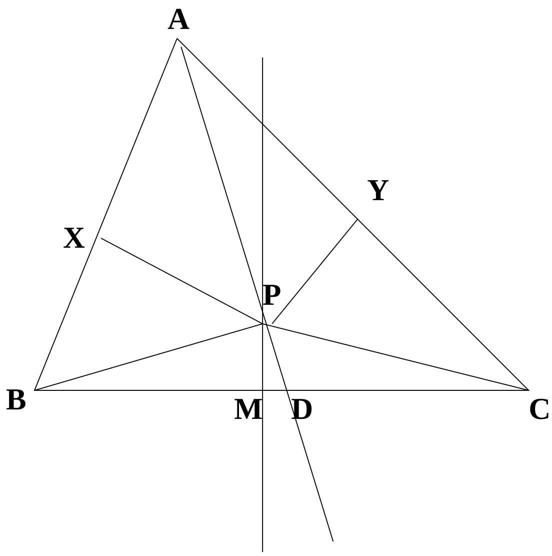
Tour 13 - All Triangles Are Equilateral

Let's carefully examine this bogus proof.



Pick any arbitrary triangle ABC.

Let the internal angle bisector of $\angle A$ and the perpendicular bisector of side BC meet at point P, as shown. Construct PB and PC, and find points X and Y on AB and AC respectively so that $PX \perp AB$ and $PY \perp AC$.

Now, $\triangle APX \cong \triangle APY$, by Angle-Side-Angle (ASA) congruency test. Hence, $PX = PY$, and $AX = AY$. Since P is the perpendicular bisector of side BC, we have $BM = MC$, so by the Side-Angle-Side congruency test, $\triangle PMB \cong \triangle PMC$, and so $PB = PC$.

Finally, $\triangle PXB \cong \triangle PYC$ by the Hypotenuse-Side congruency test, since $PX = PY$, $PB = PC$, and $\angle PXB = \angle PYC = 90°$. Thus, $XB = YC$.

We have proven that $AX = AY$ and $XB = YC$. Therefore, $AX + XB = AY + YC$, which simplifies to $AB = AC$. Since we started with an arbitrary triangle ABC, and we proved that $AB = AC$, we conclude that for any triangle ABC, two of its sides must be equal. Therefore, we have proven that all triangle are isosceles, Q.E.D.

If $\triangle ABC$ is isosceles, specifically, if $AB = AC$, then the angle bisector will be the same line as the perpendicular bisector! Here's a more rigorous justification of this claim:

Construct the line l from A to M, where M is the midpoint of BC. If we can show that $\angle AMB = 90°$, then this line l must be the perpendicular bisector of BC, by definition. Well, $BM = MC$, and $AB = AC$, thus by the Side-Side-Side congruency test, $\triangle AMB \cong \triangle AMC$. Hence $\angle AMB = \angle AMC$, and because these two angles add up to 180°, both must be 90°. Thus, this line l is the perpendicular bisector of BC. Furthermore, since $\triangle AMB \cong \triangle AMC$, we have $\angle BAM = \angle CAM$, and so AM is the angle bisector of $\angle A$. Therefore, line l is also the internal angle bisector of $\angle A$. And this proves the claim.

Since point P is defined to be the intersection point of the internal angle bisector of $\angle A$

and the perpendicular bisector of side BC, point P is undefined in this case because there are infinitely many points that lie on both lines, since they are the exactly same line!

Now suppose $\triangle ABC$ is not isosceles. In other words $AB \neq AC$. Without loss of generality, assume $AB < AC$. (So if we had $AC < AB$, the proof would be the exact same, by symmetry - so we can just assume that $AB < AC$ "without loss of generality").

Our goal is to prove that D must lie to the left of M. If we can do this, then the angle bisector AD must meet the perpendicular bisector *outside* the triangle. If D is to the right of M, clearly these two lines will meet inside the triangle, as in our bogus diagram.

Well, by the Internal Angle Bisector Theorem, we have $\frac{AB}{AC} = \frac{BD}{DC}$.

But $AB < AC$, so this tells us that $\frac{BD}{DC} = \frac{AB}{AC} < 1$. So in other words, we have $BD < DC$. Well, by definition, $BM = MC$, so if BD is less than DC, then D must be to the left of point M. Here's a more formal proof: suppose D is to the right of M on the line BC. Then $BM < BD$. However, $CM = BM < BD < DC$, so $CM < CD$. However, D is closer to the point C than M is, and that is a contradiction.

Therefore, we have proven that point D must be to the left of point M on the side BC, and so we conclude that the point P must be outside the triangle.