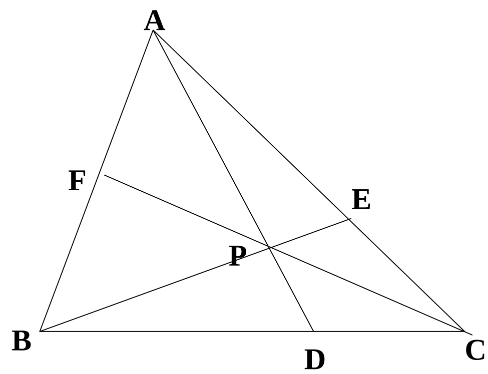
Tour 14 - Ceva's Theorem

Construct an arbitrary triangle ABC.



Pick points D, E, and F so that D is on BC, E is on AC, and F is on AB.

Ceva's Theorem states that line segments AD, BE, and CF are concurrent, i.e., they meet at a single point, if and only if

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Before we prove this incredible theorem, let us see how this is useful.

Ceva's Theorem states that provided this equation holds, the line segments AD, BE, and CF *must* be concurrent. For example, if AD, BE,and CF are *medians* of the triangle, then D, E, and F are the midpoints of their respective sides. Thus, we have $AF = FB$, $BD = DC$, and $CE = EA$.

Then, by Ceva's Theorem, we have $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1 \cdot 1 \cdot 1 = 1$.

And so, we have proven that the three medians *must* be concurrent, and we call this point of concurrency the *centroid*.

Let's do one more example. We'll prove that the three internal angle bisectors of a triangle are concurrent.

Let AD, BE, and CF be internal angle bisectors of $\angle A$, $\angle B$, and $\angle C$, respectively.

By the Internal Angle Bisector Theorem, we have $\frac{BD}{DC} = \frac{AB}{AC}$, and similarly, we have $\frac{AF}{FB} = \frac{AC}{BC}$, and $\frac{CE}{EA} = \frac{BC}{AB}$.

Therefore, we have

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{AC}{BC} \cdot \frac{AB}{AC} \cdot \frac{BC}{AB} = 1.$$

And so by Ceva's Theorem, the internal angle bisectors are concurrent. We call this point of concurrency the *incentre*.

Now, let's prove Ceva's Theorem. For convention, we'll write $(\triangle XYZ)$ to represent the *area* of $\triangle XYZ$.

We'll first prove that if the three line segments are concurrent, then $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$.

Look at $\triangle PBD$ and $\triangle PDC$. Notice that both triangles have the same height, or altitude. Hence, the ratio of their areas is the same as the ratio of their bases. In other words, $\frac{(\triangle BDP)}{(\triangle CDP)} = \frac{BD}{DC}$.

Also, we have $\frac{(\triangle BDA)}{(\triangle CDA)} = \frac{BD}{DC}$, since both of these triangles share a common altitude as well. Hence the ratio of their areas is the ratio of their bases, which is $BD : DC$.

Let $\frac{(\triangle BDA)}{(\triangle CDA)} = \frac{(\triangle BDP)}{(\triangle CDP)} = t$.

Then, $(\triangle BDA) = t(\triangle CDA)$, and $(\triangle BDP) = t(\triangle CDP)$.

Subtracting one equation from the other, we get $(\triangle BDA) - (\triangle BDP) = t(\triangle CDA) - t(\triangle CDP)$, and looking at the diagram, we see that this is equivalent to saying $(\triangle APB) = t(\triangle APC)$. In other words, $\frac{BD}{DC} = \frac{(\triangle APB)}{(\triangle APC)}$.

Also, by symmetry, we have $\frac{AF}{FB} = \frac{(\triangle APC)}{(\triangle BPC)}$ and $\frac{CE}{EA} = \frac{(\triangle BPC)}{(\triangle APB)}$.
Therefore, we have

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{(\triangle APB)}{(\triangle APC)} \cdot \frac{(\triangle APC)}{(\triangle BPC)} \cdot \frac{(\triangle BPC)}{(\triangle APB)} = 1.$$

Now we want to prove the converse. We want to prove that if line segments AD, BE, and CF satisfy $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$, then these three lines must be concurrent at some point P.

Let AD and BE meet at P. We want to show that CF also passes through point P. So suppose that CP meets the line AB at some point F'. Our goal is to show that $F' = F$, i.e., F and F' represent the same point. If we can do this, then we will have proven that the three line segments all go through point P.

We are given that $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$.

And since the line segments AD, BE, and CF' all concur at point P, by what we just proved, we know that $\frac{AF'}{F'B} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$

Comparing these two equations, we immediately know that $\frac{AF}{FB} = \frac{AF'}{F'B}$. So F and F' divide the line segment AB into equal ratios. In other words, F and F' must be the same point. This proves that AD, BE, and CF all pass through point P, and we are done.