Tour 18 - Problems Involving Circles

1. Pick any point D on major arc AB (in other words, pick D so that A, D, B, and C are vertices on the circle, in clockwise order). Then, by the Star-Trek Theorem, $\angle ADB = 50°$, since it is exactly half of $\angle AOB$. So now we know that $\angle ADB = 50°$. But opposite angles in a cyclic quadrilateral are supplementary. Thus, $\angle ADB + \angle ACB = 180°$. It follows that $\angle ACB = 130°$.

2. Since A is the centre of the big circle, we have $AB = AD$. Since AB is a diameter of the small circle, $\angle ACB = 90°$. Hence, $\angle ACD = 180° - 90° = 90°$. Thus, triangles ABC and ADC are both right-angled triangles with $AB = AD$ (and $AC = AC$). So by the Hypotenuse-Side congruency test, $\triangle ABC \cong \triangle ADC$, and from this it follows that $BC = DC$.

3. Since $OZ = OX = r$, we have $CZ = CX = r$. Thus, $AZ = b - r$ and $XB = a - r$. Since $AZ = AY$ and $BZ = BY$ (equal tangents theorem), we have $c = AB = AY + YB = AZ + XB = (b - r) + (a - r) = a + b - 2r$.

   Hence, $2r = a + b - c$, or $r = \dfrac{a+b-c}{2}$.

4. Draw a circle with centre A and let BC be a tangent to the circle. Draw a line through B and A, intersecting the circle at X and Y. Label the diagram so that X is closer to B than Y is. So if we let the radius of the circle be b (so $AC = b$), $BC = a$, and $AB = c$, we have $AX = b$ and $XB = c - b$. Furthermore, $YB = YA + AX + XB = c + b$.

   Then by the Power of a Point Theorem, we have $BC^2 = BX \cdot BY$, which becomes $a^2 = (c - b)(c + b) = c^2 - b^2$. Simplifying, we get $a^2 + b^2 = c^2$.

5. Construct E on BD such that $\angle AED = \angle ABC$. Now, $\angle ADE = \angle ADB = \angle ACB$, since $ABCD$ is a cyclic quadrilateral. Thus, by the Angle-Angle-Angle similarity test, $\triangle ADE \sim \triangle ACB$. So, $\frac{AD}{DE} = \frac{AC}{CB}$. Thus, $DE \cdot AC = AD \cdot BC$. Similarly, you can show that $\triangle ADC \sim \triangle AEB$, so $\frac{DC}{AC} = \frac{EB}{AB}$. Thus, $EB \cdot AC = AB \cdot DC$.

   Thus, $BD \cdot AC = DE \cdot AC + EB \cdot AC = AD \cdot BC + AB \cdot DC$, as required.

6. Construct a circle, and let BD and AC be diameters of the circle. So both these lines meet at the centre O. Now, all four angles of quadrilateral $ABCD$ are 90°, so $ABCD$ is a rectangle. Let $AC = BD = c$, $AB = CD = a$, and $BC = AD = b$. By Ptolemy's Theorem, we have $a \cdot a + b \cdot b = c \cdot c$, or $a^2 + b^2 = c^2$.

7. Label the centres of the circles A, B, C, and so on, going from bottom to top, left to right. Since KF is vertical, we need to show that $\angle FKL = 90°$. Similarly, we need to show that $\angle LMH = 90°$. This will prove that the fifth row is perfectly horizontal.

   The distance between the centres of the touching bottles is just the diameter of a bottle. Thus, I is equidistant from F, K, and L, making I the circumcentre of $\triangle FKL$. Now, if I were to lie on FL, then FL would be a diameter of the circumcircle of $\triangle FKL$, making $\triangle FKL$ the desired right angle. Thus, it remains only to show that I lies on FL.

At the opposite part of this figure, we know that $\triangle BCH$ is right-angled at C, and that E is its circumcentre. So E is the midpoint of BH. Now clearly the four quadrilaterals around G are rhombi, and hence are parallelograms, making sides IL, GJ, and EH parallel. Similarly, FI, DG, and BE are parallel. Therefore, FI and IL both lie in the direction of line segment BEH, making them parts of the same straight line. Thus, I lies on FL, and we are done.

Therefore, $\angle FKL = 90°$, and similarly, $\angle LMH = 90°$, so this proves that the fifth row is perfectly horizontal.