Tour Two - The Pigeonhole Principle

Consider the following assertions. It won't take you long to convince yourself that they are indeed valid:

a) Among any thirteen people, at least two are born in the same month.

b) Among any five people, at least three are of the same gender.

c) Among any twenty-nine people, at least five are born on the same day of the week.

These are illustrations of the **Pigeonhole Principle** which states that *if* $nk+1$ *pigeons are placed into* k *pigeonholes, then at least* $n+1$ *pigeons will be placed in the same pigeonhole.*

Note: n and k must be positive integers. In example a) above, we have $k = 12$ and $n = 1$. Each month is a pigeonhole.

Problem 2.1: *Pick any* 6 *elements from the set* $\{1, 2, \ldots 10\}$. *Show that you can always find two elements that add up to* 11.

Group the elements in pairs so that the elements in each pair add up to 11. We have the following 5 "pigeonholes": $(1, 10), (2, 9), (3, 8), (4, 7)$, and $(5, 6)$. By the Pigeonhole Principle, if we pick 6 elements, two elements from the same pair must be selected. These two elements will sum to 11.

Problem 2.2: *Pick any seven points or on inside a regular hexagon of side length* 1. *Show that there are two points that are at most* 1 *unit apart.*

Divide the hexagon into six equilateral triangles of side length 1. By the Pigeonhole Principle, if we select 7 points, at least two must lie in the same equilateral triangle. Since the triangle has side length 1, the farthest distance between these two points will be 1, when the points are at the vertices of the triangle. Thus there must be two points that are at most 1 unit apart.

Problem 2.3: *A lattice point is a point whose coordinates are both integers. For example,* $(19, 99)$ *is a lattice point, but* $(3, -2.5)$ *is not. Pick any five distinct lattice points. Prove that it is possible to select two of these points such that its midpoint is also a lattice point.*

Recall that the midpoint of (x_1, y_1) and (x_2, y_2) is $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.

There are four "types" of points: (odd, odd), (odd, even), (even, odd), and (even, even). By the Pigeonhole Principle, if we select five points, at least two must be of the same type. Consider these two points. The sum of both the x and the y coordinates will be even (since we are adding integers of the same parity in both cases), and hence its midpoint will be a lattice point.

Problem 2.4: *Consider six points in the plane, of which no three are collinear (i.e., no line passes through three of these points). Draw lines that connect each point to every other point (there should be* 15 *lines in all), and colour each of these lines either turquoise or burgundy. Show that there must exist a monochromatic triangle (i.e., a triangle where all* *three sides have the same colour).*

We'll proceed by contradiction. Suppose that there is no monochromatic triangle. Pick an arbitrary point, and call it P. There is an edge from P to each of the other five points. Since we are colouring each of our five edges in one of two colours, by the Pigeonhole Principle, at least three of the edges will be of the same colour. Consider three of these edges, call them PA, PB, and PC. Say the edges are coloured burgundy (doesn't really matter). If AB is coloured burgundy, then PAB is a triangle where all three sides are coloured the same, and this gives us a contradiction. Thus, AB must be coloured turquoise. Similarly, BC and AC must be coloured turquoise, otherwise PBC and PAC will be monochromatic triangles. Then triangle ABC has all three sides coloured turquoise, and so ABC is a monochromatic triangle, and this is a contradiction. Thus, no matter how we colour the edges, there must exist a triangle where all three sides have the same colour.

Problem 2.5: *Suppose there are six people at a party. Show that among the six, there are either three mutual acquaintances or three mutual strangers.*

Represent the six people as points in the plane. Draw edges joining each pair of people, so there should be 15 edges in all. For the edge joining P to Q, colour the edge turquoise if P and Q are acquaintances, and colour the edge burgundy if P and Q are strangers. Do this with every other edge. So we have six points, and fifteen lines coloured with one of two colours. This is the same as the previous question! From Problem 2.4, there must be a monochromatic triangle ABC. In the context of this problem, A, B, and C are either mutual acquaintances (if the triangle is turquoise) or they are mutual strangers (if the triangle is burgundy).

Problem 2.6: *Prove that some multiple of $\sqrt{2}$ lies within $\frac{1}{1000000}$ of an integer.*

Consider the first $999,999$ multiples of $\sqrt{2}$, namely $\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, 4\sqrt{2}, \ldots, 999,999\sqrt{2}$. We will show that one of these multiples lies within $\frac{1}{1000000}$ of an integer.

Let's look at the first six digits after the decimal point of each number:

$$\begin{aligned}
\sqrt{2} &= 1.\mathbf{414213}562\ldots \\
2\sqrt{2} &= 2.\mathbf{828427}124\ldots \\
3\sqrt{2} &= 4.\mathbf{242640}686\ldots \\
&\vdots \\
999998\sqrt{2} &= 1414210.\mathbf{733945}970\ldots \\
999999\sqrt{2} &= 1414212.\mathbf{148159}532\ldots
\end{aligned}$$

There are one million possibilities for these six digits, namely 000000,000001,000002, $\ldots, 999998, 999999$. If one of these multiples has the first six digits after the decimal point being either 000000 or 999999, then this multiple will be within one-millionth of an integer and we will be done. So assume this is not the case. We will derive a contradiction.

Then there are only $999,998$ possibilities for the first six digits after the decimal point. But we have $999,999$ multiples of $\sqrt{2}$, and so by the Pigeonhole Principle, at least two of these multiples of $\sqrt{2}$ will have its first six digits after the decimal point being the same. Thus for some positive integers j and k, with $1 \leq j < k \leq 999999$, we have $j\sqrt{2}$ and $k\sqrt{2}$ having the same first six digits after the decimal point. Then $(k-j)\sqrt{2}$ has its first six digits after the decimal point being either 000000 or 999999. Because $k-j$ is between 1 and 999999, we have found a contradiction - there is a multiple of $\sqrt{2}$ in our list that has its first six digits being 000000 or 999999.

Thus, there must be some multiple of $\sqrt{2}$ that lies within $\frac{1}{1000000}$ of an integer.

Here is an additional problem.

Problem 2.7: *Show that there must exist an integer containing only the digits* 0 *and* 1 *which is divisible by* 1999.

Let S be the first 2000 numbers in the set $1, 11, 111, 1111, \ldots$. 9 Look at the remainder of each of these numbers when divided by 1999. There are 1999 possible remainders, from 0 to 1998. So by the Pigeonhole Principle, two of these numbers must have the same remainder when divided by 1999. Let these two numbers be a and b, where $a > b$. Then $a - b$ is divisible by 1999, and this number only contains 0's and 1's, because we are subtracting a string of ones from another string of ones. (Example: $111111 - 1111 = 110000$). Thus, we have shown that there must exist an integer containing only the digits 0 and 1 which is divisible by 1999.