Solution To Pathfinder Problem

Let P_n be the probability of Ryan winning the game if he makes exactly n mistakes. He is allowed to make up to three stepping mistakes, so we are interested in P_0, P_1, P_2, and P_3.

$P_0 = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{72}$, since he must make the four correct moves (up, right, right, up) in a row.

On his first move there are four directions he can travel. So he has a one-fourth chance of moving to the correct square, and a three-fourths chance make a mistake. If he gets his first move wrong, then he has eliminated one incorrect square, so his chances of moving to the correct square now becomes one-third, and his chances of once again moving to an incorrect square now becomes two-thirds. And so on.

So the probability that Ryan moves to the correct square after zero mistakes is $\frac{1}{4}$, after one mistake is $\frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$, after two mistakes is $\frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{4}$ and after three mistakes is $\frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{4}$. In all cases, the probability that Ryan moves from the 1 to the 5 is $\frac{1}{4}$. If we pretend (for now) that Ryan can make as many mistakes as he wants, he will obviously make the correct move eventually. We will take into account the fact that he can make at most three mistakes later.

Similarly, the probabilities do not change for the other moves, i.e., he has a one-third chance of moving from the 5 to the 6, regardless of how many mistakes he makes. So to calculate P_1, we just have to examine all the different ways he can make one mistake, and each of these different ways occurs with probability $\frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{72} = P_0$.

Ryan has to make four correct moves to win the truck, and call these moves 1, 2, 3, and 4, respectively. Thus, he can make his mistake at any of these four moves, and so $P_1 = 4P_0 = \frac{4}{72}$.

If Ryan can make two mistakes, he can make both mistakes on the same move, or make a mistake on two separate moves. His possibilities for making two mistakes are: $(1,1)$, $(2,2)$, $(3,3)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,3)$, $(2,4)$, and $(3,4)$. Note that $(4,4)$ is not a possibility because there are only two possible choices for the fourth and final move, and so Ryan can make at most one error on the last step. So there are nine possible ways for him to make two mistakes, and so $P_2 = 9P_0 = \frac{9}{72}$.

And here are the possible ways Ryan can make three mistakes. There are fourteen of them:

$(1,1,1)$, $(1,1,2)$, $(1,1,3)$, $(1,1,4)$, $(1,2,2)$, $(1,2,3)$, $(1,2,4)$, $(1,3,3)$, $(1,3,4)$, $(2,2,3)$, $(2,2,4)$, $(2,3,3)$, $(2,3,4)$, and $(3,3,4)$. It follows that $P_3 = 14P_0 = \frac{14}{72}$.

Thus, we have shown that $P_0 = \frac{1}{72}$, $P_1 = \frac{4}{72}$, $P_2 = \frac{9}{72}$, and $P_3 = \frac{14}{72}$. Now, let's look at the pricing games. If Ryan makes a mistake, he has to correctly answer a pricing game before he can make another turn.

If he makes one mistake, he has to win one pricing game. The only way this cannot happen is if Ryan gets all three pricing games wrong, which occurs with probability $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. So the probability that Ryan wins one pricing game is $1 - \frac{1}{8} = \frac{7}{8}$.

If Ryan makes two mistakes, he has to win two pricing games. So he can either get the first two games right (probability $\frac{1}{4}$), lose the first and win the next two (probability $\frac{1}{8}$), or win the first, lose the second, and win the third (probability $\frac{1}{8}$). And so the probability that

Ryan wins two pricing games is $\frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$.

Finally, if Ryan makes three mistakes, he must win all three pricing games (like he did). This occurs with probability $\frac{1}{8}$.

Therefore, the probability that Ryan wins the Dodge Dakota is:

$$\begin{aligned}
P &= P_0 + \frac{7}{8}P_1 + \frac{1}{2}P_2 + \frac{1}{8}P_3 \\
&= P_0 + \frac{7}{8} \cdot 4P_0 + \frac{1}{2} \cdot 9P_0 + \frac{1}{8} \cdot 14P_0 \\
&= P_0(1 + \frac{7}{2} + \frac{9}{2} + \frac{7}{4}) \\
&= P_0(1 + 8 + \frac{7}{4}) \\
&= P_0(\frac{43}{4}) \\
&= \frac{1}{72} \cdot \frac{43}{4} \\
&= \frac{43}{288}
\end{aligned}$$