Tour 22 - Binomial Coefficients

Problem 22.1: *Prove that* $\binom{n}{k} = \binom{n}{n-k}$, *for all positive integers* n *and* k *with* $k \leq n$.

We can prove this very easily algebraically:

Since $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!}$, we have the desired result.

But let's also do it combinatorially. The number $\frac{n!}{k!(n-k)!}$ represents the number of permutations of the word containing k A's and $(n-k)$ B's. And the number $\binom{n}{n-k}$ represents the number of permutations of the word containing $(n-k)$ A's and k B's. By symmetry these two quantities must be equal, so we conclude that $\binom{n}{k} = \binom{n}{n-k}$.

Problem 22.2: *Prove that for all positive integers* r, s, *and* n, *we have:*

$$\binom{r}{0}\binom{s}{n} + \binom{r}{1}\binom{s}{n-1} + \ldots + \binom{r}{n-1}\binom{s}{1} + \binom{r}{n}\binom{s}{0} = \binom{r+s}{n}.$$

Consider a room of r boys and s girls, and say we want to pick n people from this group to form a committee. Well, there are $\binom{r+s}{n}$ ways to do that. So that explains the right side. But can we also represent this total in a different way?

Well, if we had 0 boys on our committee, we'd have to have n girls, and so the number of ways of having a committee of 0 boys and n girls is $\binom{r}{0}\binom{s}{n}$. If we had 1 boy and $n-1$ girls, there would be $\binom{r}{1}\binom{s}{n-1}$ possible committees. And we keep doing this, considering the case when we have 2 boys, 3 boys, and so on, up to n boys. So adding up all these cases, we get

$$\binom{r}{0}\binom{s}{n} + \binom{r}{1}\binom{s}{n-1} + \ldots + \binom{r}{n-1}\binom{s}{1} + \binom{r}{n}\binom{s}{0}.$$

And this sum represents the total number of possible committees we can form with n people, from a group of r boys and s girls. But we agreed that this number was $\binom{r+s}{n}$. So the left side equals the right side, and we are done.

Problem 22.3: *The following is known as Pascal's Identity:*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

(a) Find a combinatorial proof of Pascal's Identity.

(b) Explain how this formula relates to Pascal's Triangle.

This is a very nice proof. How many ways are there of picking a committee of k people from a group of n people? Well, the answer is by definition $\binom{n}{k}$. Now, let's pick one of those n people, say her name is Chantel.

If Chantel *is* on the committee, then there are $n-1$ people left, and we need to choose $k-1$ people from the remaining people to form our committee. And this can be done in $\binom{n-1}{k-1}$ ways.

If Chantel *is not* on the committee, then there are $n-1$ people left, and we need to choose k people from the remaining people to form our committee. And this can be done in $\binom{n-1}{k}$ ways.

Since Chantel is either on the committee, or not on the committee, we've considered all possible cases. Thus, the number of possible committees we can form with k people from a group of n people is $\binom{n-1}{k-1} + \binom{n-1}{k}$. But we agreed earlier that this number was also equal to $\binom{n}{k}$. And so we're done.

Problem 22.4: *The Binomial Theorem states that for any positive integer n,*

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n$$

(a) Prove the Binomial Theorem using mathematical induction.

(b) Prove that the sum of the elements in the n^{th} row of Pascal's Triangle is 2^n.

(c) Prove that the alternating sum of the elements in the n^{th} row of Pascal's Triangle is 0. For example, in the fourth row, we have $1-4+6-4+1=0$.

For a proof of the Binomial Theorem using mathematical induction, please see my article entitled "Riveting Properties of Pascal's Triangle", published in Crux Mathematicorum with Mathematical Mayhem, sometime in 1998. I handed out copies of this article in class – if you weren't there that day, please see me so I can print you another copy.

Substitute $a=1$ and $b=1$ into the Binomial Theorem. Then, we have

$$\begin{aligned}(1+1)^n &= \binom{n}{0}1^n 1^0 + \binom{n}{1}1^{n-1}1^1 + \binom{n}{2}1^{n-2}1^2 + \ldots + \binom{n}{n-1}1^1 1^{n-1} + \binom{n}{n}1^0 1^n \\ &= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}\end{aligned}$$

Thus, 2^n is the sum of the elements of the n^{th} row of Pascal's Triangle, as required.

Substitute $a=1$ and $b=-1$ into the Binomial Theorem. Then, we have

$$\begin{aligned}(1-1)^n &= \binom{n}{0}1^n(-1)^0 + \binom{n}{1}1^{n-1}(-1)^1 + \binom{n}{2}1^{n-2}(-1)^2 + \ldots + \binom{n}{n}1^0(-1)^n \\ &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n}\end{aligned}$$

Thus, the alternating sum of the elements of the n^{th} row of Pascal's Triangle is always 0.

We can substitute anything for a and b. So for example, if we wanted to find an expression for $(5-\sqrt{2})^6$, or something nasty like that, we could do that by substituting $a = 5$, $b = -\sqrt{2}$, and $n = 6$ into our Binomial Theorem Formula, and evaluating the expression.

Problem 22.7:

(a) Determine the number of solutions (a_1, a_2, a_3) in positive integers to the equation $a_1 + a_2 + a_3 = 10$.

(b) Determine the number of solutions (a_1, a_2, a_3) in non-negative integers to the equation $a_1 + a_2 + a_3 = 7$.

What do you notice about these two answers? Is this a coincidence?

Look at the ten-letter string $OOOOOOOOOO$, and let's insert two X's in this string so that each X is sandwiched between two O's to form a new string. For example, one possible "new string" is $OOOXOXOOOOOO$. We can associate each of these new strings with an ordered triplet (a_1, a_2, a_3), where a_1 is the number of O's before the first X, a_2 is the number of O's between the two X's, and a_3 is the number of O's after the second X. So our permutation corresponds to $(3, 1, 6)$. Note that each new string corresponds to a unique triplet (a_1, a_2, a_3) in positive integers to the above equation, *and* vice-versa. In other words, we are establishing a one-to-one correspondence (a bijection) between the set of solutions in positive integers to $a_1 + a_2 + a_3 = 10$ and the set of strings of two X's and ten O's so that each X is sandwiched between two O's. So if we can find the number of elements in the latter set, then that would equal the number of elements in the former set.

Well, there are $\binom{9}{2} = 36$ elements in the latter set, since we have to insert two X's somewhere in the nine "break points" (i.e., between consecutive O's). So it follows that there are exactly 36 solutions in positive integers to the equation $a_1 + a_2 + a_3 = 10$.

Here is a clever way to tackle part b) of this question. Consider all of the permutations of 7 O's and 2 X's. For example, two such permutations are $OXOOOXOOO$ and $OOOXXOOOO$. We can associate each permutation with an ordered triplet (a_1, a_2, a_3) where a_1 is the number of O's before the first X, a_2 is the number of O's between the two X's, and a_3 is the number of O's after the second X. So in our examples, the first permutation corresponds to $(1, 3, 3)$ and the second corresponds to $(3, 0, 4)$. Note that each permutation corresponds to a unique triplet (a_1, a_2, a_3) in non-negative integers, *and* vice-versa. So once again, we are establishing a one-to-one correspondence between the set of non-negative solutions to $a_1 + a_2 + a_3 = 7$ and the set of permutations of seven O's and two X's. Clearly there are $\binom{9}{2} = 36$ elements in the latter set. Thus, there are 36 solutions in non-negative integers to the equation $a_1 + a_2 + a_3 = 7$.

It is no coincidence that these two answers are equal. To prove that, all we need to do is establish a one-to-one correspondence between the 36 solutions in part a) and the 36 solutions in part b). Here is such a correspondence: Pair the set (x, y, z) in part a) with $(x-1, y-1, z-1)$ in part b). Note that (x, y, z) is a solution in positive integers to $x + y + z = 10$ if and only if $(x-1, y-1, z-1)$ is a solution in non-negative integers to $(x-1) + (y-1) + (z-1) = 7$. So the number of solutions in both sets must be the same. Thus, if there are 36 solutions to part a), then we conclude that there must be exactly 36 solutions to part b) as well.

Problem 22.9: *Prove this incredible fact: if the binary representation of n contains p ones, then there are 2^p odd numbers in the n^{th} row of Pascal's Triangle.*

(For example, $13 = 1101_2$, so there are $2^3 = 8$ odd numbers in the thirteenth row of Pascal's Triangle).

Please see my article on Pascal's Triangle for a solution to this problem.