Tour Three - Symmetry

Here are the solutions to the problems we discussed in class today. It is truly amazing that we can take this simple idea of symmetry and use it to solve so many difficult problems.

Problem 3.1: *Weighing the baby at the clinic was a problem. The baby would not keep still and caused the scale to wobble. So I held the baby and stood on the scales while the nurse read off* 185 *pounds. Then the nurse held the baby while I read off* 145 *pounds. Finally, I held the nurse while the baby read off* 310 *pounds. What was the combined weight of the nurse, baby, and me?*

Essentially this is a simple system of equations. We wish to determine the value of $x + y + z$, where $x + z = 185$, $y + z = 145$, and $x + y = 310$. Instead of solving for each unknown, look at the symmetry of the three equations, and use it to your advantage.

Adding the three equations, we have $2x+2y+2z = 640$, and dividing by 2, we determine that the combined weight was 320 pounds. Notice we didn't have to solve for x, y, and z!

Problem 3.2:

Eve and Oddie play the following game. One of them flips 2001 *coins and Eve wins if the number of heads is even, and Oddie wins if the number of heads is odd. What is the probability that Eve will win the game?*

This is a great question, given to me by Peter Taylor of Queen's University, in a course he taught me. The title of the course was "Mathematics and Poetry". Here I present three of the five solutions I know (the other two involve Pascal's Triangle and are not very elegant).

Solution 1: The probability of getting an even number of heads is the same as the probability of getting an even number of tails. This is because the situation with regards to heads and tails is entirely symmetric, and so we must have P(even number of heads) = P(even number of tails).

But there are 2001 coins, so you get an even number of tails if and only if you get an odd number of heads (since 2001 is odd). Thus, P(even number of tails) = P(odd number of heads).

So we have shown that P(even number of heads) = P(odd number of heads), and so the probability of each scenario is $\frac{1}{2}$, and so the probability that Eve will win the game is $\frac{1}{2}$, or fifty percent. Isn't that a terrific argument?

Solution 2: Write down each of the 2^{2001} possible outcomes. For example, if there are 3 coins, the 2^3 possible outcomes are HHH, HHT, HTH, HTT, THH, THT, TTH, and TTT. Associate to each outcome a "partner" obtained by replacing every H by T and every T by H, and form your pairs this in this manner. So in our three-coin example, our pairs are (HHH, TTT), (HHT, TTH), (HTH, THT), and (HTT, THH). Then for each pair, one of the outcomes has an even number of heads, and the other has an odd number of heads. The two outcomes must have a different parity because the total number of coins is 2001, and so there are a total of 2001 H's in each pair. Now the set of all outcomes is partitioned as a collection of even-odd pairs, and so there are just as many outcomes with an even number of heads as outcomes with an odd number of heads, and so Eve must have a fifty percent chance of winning this game.

Solution 3: Pick one of the coins and colour it red. Now toss the coins, look at the outcome, then pick up the red coin and turn it over. What we get is a different outcome with the *opposite parity*. That is, if the first outcome had an even number of heads, then the second one must have an odd number, and vice-versa. This device (changing the state of the red coin) gives us a simple way of partitioning the set of outcomes into pairs with opposite parities. It follows that there must be the same number of outcomes with even and odd numbers of heads, and so Eve must have a fifty percent chance of winning this game.