Tour 5 - Mathematical Induction

A very powerful problem-solving tool is *Mathematical Induction*.

To prove that a statement is true for all positive integers n, it suffices to do the following:

1) Prove the statement for $n = 1$.

2) Prove that *if* the statement is true for $n = k$, *then* the statement must also be true for $n = k + 1$.

For a more formal introduction to mathematical induction, check out

$$http://www.cs.umd.edu/\ smith/induction/$$

Example Problem: *Show that if n is a positive integer, then $n^2 + n + 1$ is odd.*

Let $f(n) = n^2 + n + 1$. We shall verify both parts of our induction proof: i) that $f(1)$ is odd, and ii) $f(k)$ is odd implies that $f(k + 1)$ is odd.

Note that $f(k + 1) - f(k) = [(k + 1)^2 + (k + 1) + 1] - [k^2 + k + 1] = 2k + 2 = 2(k + 1)$, which is an even number. Therefore, $f(k)$ and $f(k+1)$ must have the same parity for all k. Hence, if $f(k)$ is odd, then that implies that $f(k + 1)$ is odd, and this is true for all positive integers k.

Since $f(1) = 3$ is an odd number, by mathematical induction we conclude that $f(n)$ is odd for all positive integers n.

Problem 5.1 Prove that the sum of the interior angles of any n-gon is $180(n-2)$ degrees, where $n \geq 3$.

Well, we know that any triangle has 180 degrees, so the statement is clearly true for $n = 3$.

Let $P(k)$ be the sum of the interior angles of any k-gon. We want to prove that *if* $P(k) = 180(k - 2)$, *then* $P(k+1) = 180(k - 1)$. Well, take any $k+1$-gon. We can subdivide this $k+1$-gon into a k-gon and one triangle. By our induction hypothesis, the k-gon has the sum of its interior angles being $180(k - 2)$. And the angles of our triangle add up to 180. Therefore, we must have $P(k + 1) = 180(k - 2) + 180 = 180(k - 1)$. Hence, we've proven that *if* $P(k) = 180(k - 2)$, *then* $P(k + 1) = 180(k - 1)$, and that completes the proof.

Problem 5.2 On a large flat field, 2001 people are positioned so that for each person the distances to all the other people are different. Each person holds a water pistol and at a given signal fires and hits the person who is closest. Show that there is at least one person who will be left dry.

We shall prove by mathematical induction that if there are n people, where n is an odd number larger than 3, then there is at least one person who will be left dry. Note that this conclusion does *not* hold for any even integer n - see if you can justify why that is true.

First we shall show that the statement holds for $n = 3$.

Look at the pairs of distances between three people, positioned anywhere on the field. So if the people are A, B, and C, look at the distances from A to B, B to C, and C to A. Since all of these distances are different, one of these distances is smaller than the other two. Suppose the shortest distance is the distance from A to B. Then A will shoot B, and B will shoot A. Regardless of what C does, she will be left dry. So for $n = 3$, at least one person will be left dry.

Now suppose that we have proven the statement for $n = 2k + 1$, where k is a positive integer. So if there are $2k + 1$ people positioned anywhere on the field, at least one will be left dry after every person has fired a shot. We shall show the statement is true for the next odd number. Position $n = 2k + 3$ people anywhere on the field. We shall show that when these $2k + 3$ people fire their shots, at least one of them will be left dry.

Look at all the distances between two people, and select the shortest distance between two people. (If there is a tie between pairs, just pick one of them). Let these two people be A and B. So A is closer to B than any other person, and B is closer to A than any other person. So A will shoot B, and B will shoot A. Now we can completely ignore A and B, because they will shoot each other and not anybody else. So consider the other $2k+1$ people on the field. Since each fires a shot, there are a total of $2k + 1$ shots. If any of these people shoot A or B, there will be at most $2k$ people from the group of $2k + 1$ people who are hit, and so by the Pigeonhole Principle at least one will be left dry. Otherwise, we have $2k + 1$ people who shoot each other, and by the induction hypothesis, one of these people must be left dry. Hence, in any scenario, we have proven that if there are $2k + 3$ people, one of them will be left dry after everyone has fired a shot.

Therefore, we have proven by mathematical induction that if n is odd, then the desired conclusion holds. Since 2001 is odd, we have proven that if there are 2001 people on the field, then one of them must be left dry after everybody has fired a shot.

Problem 5.3 There are g girls and b boys playing Frogger.

If there are exactly g girls and 1 boy (i.e. $b = 1$), how many moves are required to complete the game?

We'll prove that we require exactly $2g + 1$ moves, using mathematical induction. *Note: since we agreed that we can't go backwards, there is a fixed number of moves that will be required to complete the game. So if the game can be solved in* 7 *moves, well it can't also be solved in* 9 *moves, or* 14 *moves, or whatever. The number of moves is unique.*

If $g = 1$, clearly we require three moves. Girl slides over, boy jumps over, girl slides over. So the conclusion is true for $g = 1$.

Now we'll prove that if the statement is true for $g = k$, then it must also be true for $g = k + 1$. Let's look at the case $g = k + 1$. Well, in order for $k + 1$ girls to slide over, obviously we need the first k girls to slide over. From our induction hypothesis, we require $2k + 1$ moves for the first k girls and the one boy to switch positions. So after $2k + 1$ moves, we will have k girls correctly positioned, then one empty chair, then the one boy, and finally, the last girl. Now, to finish the game all we have to do is have the girl jump over, then the boy to slide over. And thus, using our induction hypothesis, we've proven that the case $g = k + 1$ requires $2(k + 1) + 1$ moves, and so by induction we are done.