Tour Eight - Diophantine Equations

Problem 8.1: *Determine all solutions (x, y) in integers to the equation $xy = y^2 + 3$.*

Solution 1: It is clear that there is no solution when $y = 0$. Thus, we can divide both sides of the equation by y. This reduces our equation to $x = y + \frac{3}{y}$. Since x and y are both integers, that implies that $\frac{3}{y}$ must also be an integer. Hence, the only possible values for y are $y = 1, 3, -1, -3$. If $y = 1$, we have $x = 1 + \frac{3}{1} = 4$. Similarly, we find all the other solutions. There are four solutions in total: $(x, y) = (4, 1), (4, 3), (-4, -1), (-4, -3)$.

Solution 2: From $xy - y^2 = 3$, we have $y(x - y) = 3$. Now the product of y and $x - y$ is 3, so there are only four possible values for y: we can have $y = 1$, 3, -1, or -3. As we did in our previous solution, we take each value of y and find the corresponding value of x and this gives us the same four solutions as above.

Problem 8.2: *Determine all integers x for which $(x + 5)(x - 5)$ is a perfect square.*

To make this as a Diophantine Equation, we let $(x + 5)(x - 5) = N^2$, where N is some non-negative integer. This makes sense because we want $(x+5)(x-5)$ to be a perfect square. Now, $x^2 - 25 = N^2$ implies that $(x + N)(x - N) = x^2 - N^2 = 25$. Thus the integers $x + N$ and $x - N$ multiply to 25. Since $N \geq 0$, we have $x + N \geq x - N$. So we can ignore half of our cases. Our possible values for $x + N$ and $x - N$ are:

x + N	25	5	-5	-1
x - N	1	5	-5	-25

For each column (e.g., $x + N = 25$ and $x - N = 1$), we can add the two equations and divide by 2 to get x. Thus, the first column gives us $2x = (x + N) + (x - N) = 25 + 1 = 26$, and so $x = 13$. Similarly, the other three columns give us $x = 5$, $x = -5$, and $x = -13$. Hence, these are the only four possible solutions for x, and checking, we see that all four of these integers make $(x + 5)(x - 5)$ a perfect square.

Problem 8.3: *Show that a number of the form p^n, where p is prime, can never be perfect.*

Suppose on the contrary that p^n is perfect for some integers p and n, where $n \geq 1$. Then by the definition of a perfect number,

$$\begin{aligned}
p^n &= 1 + p + p^2 + p^3 + \ldots + p^{n-1} \\
p^n &= \frac{p^n - 1}{p - 1} \\
p^n(p - 1) &= p^n - 1 \\
p^{n+1} - p^n &= p^n - 1 \\
p^{n+1} - 2p^n &= -1 \\
p^n(p - 2) &= -1
\end{aligned}$$

Note: $1+p+p^2+p^3+\ldots+p^{n-1} = \frac{p^n-1}{p-1}$ *by the summation formula for a geometric series.*

If $p^n(p-2) = -1$, then p^n must be 1 or -1. However, p is prime, and $n \geq 1$, so p^n must be greater than 1, which is a contradiction. Thus, p^n cannot be perfect for any prime p and for any positive integer n.

Problem 8.4: *Determine all solutions* (x, y) *in integers to the equation* $\frac{1}{x} + \frac{1}{y} = \frac{1}{6}$.

Solution 1: Multiplying both sides by $6xy$, we get:

$$\begin{aligned} 6y + 6x &= xy \\ xy - 6x - 6y &= 0 \\ xy - 6x - 6y + 36 &= 36 \\ x(y-6) - 6(y-6) &= 36 \\ (x-6)(y-6) &= 36 \end{aligned}$$

Thus, the integers $x - 6$ and $y - 6$ must multiply to 36. We can make a table for all possible values for both $x - 6$ and $y - 6$:

x - 6	1	2	3	4	6	9	12	18	36
y - 6	36	18	12	9	6	4	3	2	1

And we can't forget the negative solutions either: e.g. $x - 6 = -1, y - 6 = -36$. Now we just read off all solutions (x, y). They are eighteen divisors of 36, including the negatives, so we should have eighteen solutions. However, notice that $(x, y) = (0, 0)$ is *not* a solution because that does not satisfy the original equation. However, the other seventeen solutions do. Hence, there are seventeen solutions in total.

Query: what happens if we changed 6 **to some other number, say** 2000**? How many solutions** (x, y) **would there be?**

Solution 2: The factoring trick above was quite sneaky. Let's solve it another way. Let's just solve for one of the variables:

$$\begin{aligned} 6y + 6x &= xy \\ xy - 6x - 6y &= 0 \\ y(x-6) &= 6x \\ y &= \frac{6x}{x-6} \\ y &= 6 + \frac{36}{x-6} \end{aligned}$$

Since y and 6 are both integers, $\frac{36}{x-6}$ must be an integer. Thus, $x - 6$ must divide 36, so we have eighteen different possibilities. $x - 4$ can equal any of: $1, 2, 3, 4, 6, 9, 12, 18, 36,$ $-1, -2, -3, -4, -6, -9, -12, -18, -36$. As before, we can now substitute each value of x to find the corresponding value for y, and this will give us all of the solutions. However, once again if $x - 6 = -6$, then $x = 0$, and $y = 0$, and this is not a solution to the original equation. So once again, we have exactly seventeen solutions.

Problem 8.5: *Determine all positive integers n for which $n^2 - 19n + 99$ is a perfect square.*

We wish to find all integers n and k satisfying the Diophantine Equation $n^2 - 19n + 99 = k^2$, where $k \geq 0$. We are going to complete the square and we will be able to write this nasty equation as a simple difference of squares.

$$\begin{aligned} n^2 - 19n + 99 &= k^2 \\ 4n^2 - 76n + 396 &= 4k^2 \\ (4n^2 - 76n + 361) + 35 &= 4k^2 \\ (2n - 19)^2 + 35 &= (2k)^2 \\ (2k)^2 - (2n - 19)^2 &= 35 \\ (2k + 2n - 19)(2k - 2n + 19) &= 35 \end{aligned}$$

Thus, $(2k + 2n - 19)$ and $(2k - 2n + 19)$ are two integers that multiply to give 35. Both of these integers must be positive, for if they were both negative, their sum would be $(2k + 2n - 19) + (2k - 2n + 19) = 4k < 0$, which would contradict the fact that $k \geq 0$. So we only need to consider positive values of $(2k + 2n - 19)$ and $(2k - 2n + 19)$. We can make a table:

2k + 2n - 19	35	7	5	1
2k - 2n + 19	1	5	7	35

Solving each of the four cases separately, we find that $(k, n) = (9, 18), (3, 10), (3, 9), (9, 1)$. Thus there are only four positive integers n that satisfy the given conditions, namely $n = 1, 9, 10, 18$.

Problem 8.6: *Find four solutions in positive integers to the equation $x^2 - 3y^2 = 1$.*

It is not too difficult to see that $(x, y) = (2, 1)$ is a solution to this equation. Thus, $2^2 - 3 \cdot 1^2 = 1$. Although it seems contrived, we are going to write this as a difference of squares, namely $(2 + 1\sqrt{3})(2 - 1\sqrt{3}) = 1$.

The idea is to try to find integers a and b so that $(a + b\sqrt{3})(a - b\sqrt{3}) = 1$. For if we can do that, we would have $a^2 - 3b^2 = 1$, and so $(x, y) = (a, b)$ would be a solution to the given equation.

From $(2 + 1\sqrt{3})(2 - 1\sqrt{3}) = 1$, we can square both sides to get:

$$\begin{aligned} (2 + 1\sqrt{3})(2 - 1\sqrt{3}) &= 1 \\ (2 + 1\sqrt{3})^2(2 - 1\sqrt{3})^2 &= 1 \\ (2 + 1\sqrt{3})(2 + 1\sqrt{3})(2 - 1\sqrt{3})(2 - 1\sqrt{3}) &= 1 \\ (7 + 4\sqrt{3})(7 - 4\sqrt{3}) &= 1 \\ 7^2 - 3 \cdot 4^2 &= 1 \end{aligned}$$

This proves that $(x, y) = (7, 4)$ is another solution to this equation.

Let's do this again. Since $(7+4\sqrt{3})(7-4\sqrt{3}) = 1$ and $(2+1\sqrt{3})(2-1\sqrt{3}) = 1$, we can multiply both equations together to get:

$$\begin{aligned}
(7+4\sqrt{3})(7-4\sqrt{3})(2+1\sqrt{3})(2-1\sqrt{3}) &= 1\cdot 1 \\
(7+4\sqrt{3})(2+1\sqrt{3})(7-4\sqrt{3})(2-1\sqrt{3}) &= 1 \\
(14+7\sqrt{3}+8\sqrt{3}+12)(14-7\sqrt{3}-8\sqrt{3}+12) &= 1 \\
(26+15\sqrt{3})(26-15\sqrt{3}) &= 1 \\
26^2 - 3\cdot 15^2 &= 1
\end{aligned}$$

This proves that $(x,y) = (26,15)$ is another solution to this equation.

Finally, if we multiply $(26+15\sqrt{3})(26-15\sqrt{3}) = 1$ and $(2+1\sqrt{3})(2-1\sqrt{3}) = 1$, we find that $(97+56\sqrt{3})(97-56\sqrt{3}) = 1$, and so $97^2 - 3\cdot 56^2 = 1$. Hence, $(x,y) = (97,56)$ is yet another solution to this equation.

Thus, we have found four solutions: $(x,y) = (2,1), (7,4), (26,15), (97,56)$.

We have just solved a special type of Diophantine equation and illustrated the general technique of solving such equations.

A **Pell's Equation** is of the form $x^2 - Dy^2 = N$, where N is an integer and D is a positive integer that is *not* a perfect square. In our problem above, $D = 3$ and $N = 1$.

We need to find one solution to $x^2 - Dy^2 = 1$, and often this can be done just by inspection. We say that the *fundamental solution* of the equation is the smallest ordered pair (a,b) satisfying the equation, where a and b are both positive integers. So in the above example, the fundamental solution is $(2,1)$. So once we have one solution to the equation $x^2 - Dy^2 = N$, we can generate infinitely many more by repeatedly multiplying $(x+y\sqrt{D})(x-y\sqrt{D}) = N$ by $(a+b\sqrt{D})(a-b\sqrt{D}) = 1$. Although we won't prove it here, this method generates *all* solutions to the Pell's Equation.

The Pell's Equation $x^2 - Dy^2 = 1$ always has a solution, although the proof of this statement is beyond the scope of this course. Hence this equation always has a fundamental solution. So for any given Pell's Equation $x^2 - Dy^2 = N$, either there are infinitely many solutions or there are no solutions. If $x^2 - Dy^2 = N$ has one solution, then it has infinitely many.