

An Analysis of the Game of Frogs

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In this informal and interactive session, we will carefully investigate a mathematical game called “FROGS” and unpack the mathematics. In the process, we will discover many surprising connections between various areas of mathematics, and develop new ways to teach topics such as the following:

1. Observing Patterns and Relationships (Gr. 7 to 9)
2. Quadratic Functions (Gr. 10 and 11)
3. Difference Sequences (Gr. 10 and 11)
4. Mathematical Induction (Gr. 12)
5. Sums of Series (Gr. 12)

To play FROGS online, please see

<http://www.brocku.ca/mathematics/learningtools/learningobjects/>

“Many students find parts of formal mathematics arbitrary and confusing. This may well be because they most frequently see only one feature of a problem. There is, however, something fundamentally false about this approach. Authentic problems are not one-dimensional; they have a number of mathematical features. It is from the mathematization of authentic problems, that is, from the detailed investigation of the extent to which various mathematical concepts may be embedded in problems, that we learn both about problems and about mathematics”.

- William Higginson, *Mathematizing FROGS: Heuristics, Proof, and Generalization in the Context of a Recreational Problem*, *The Mathematics Teacher*, 74 (1981), 505-515.

Mathematical tasks are not to be chosen lightly. Instead, tasks should be chosen because they have the potential to engage students’ intellect; can be approached in more than one interesting way; and stimulate students to make connections and develop a coherent framework for mathematical ideas.

- Professional Standards for Teaching Mathematics (NCTM, 1991)

We have three boys and three girls, seated in a row, with one chair in the middle.



Here are the rules:

- (a) Boys may only move to the right and girls may only move to the left.
- (b) Every move is either a Slide to the adjacent square, or a Jump over one position into the empty square.
- (c) Boys can only jump over girls, and girls can only jump over boys. No same-sex jumping!
- (d) The object of the game is to move all the boys to the right, and all the girls to the left, i.e., to switch positions.

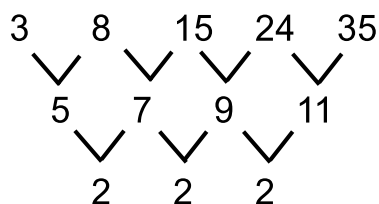
We find that 15 moves are required to complete the game.

How about other cases? What if there are n boys and n girls? We make a table of how many moves are required to complete the game for the first five values of n .

Boys	Girls	Number of Moves
1	1	3
2	2	8
3	3	15
4	4	24
5	5	35

What's the pattern? There are two formulas that emerge: $n(n + 2)$ and $(n + 1)^2 - 1$. We can quickly verify that these two expressions are equivalent.

Also, we can use the method of *difference sequences* to analyze and investigate the patterns, and conclude that the general formula must be a quadratic.



Let's look at the actual sequence of moves.

Boys	Girls	Move Sequence
1	1	SJS
2	2	SJSJJSJS
3	3	SJSJJSJJJSJJSJS
4	4	SJSJJSJJJSJJJSJJJSJJSJS
5	5	SJSJJSJJJSJJJSJJJSJJJSJJJSJJSJS

Let's look at the final sequence of moves.

Notice the palindromic sequence: $SJ^1SJ^2SJ^3SJ^4SJ^5SJ^4SJ^3SJ^2SJ^1S$.

What do you think the move sequence will be if we had n boys and n girls?

It appears that the number of slides required will be $2n$ and the number of jumps is $1 + 2 + 3 + \dots + (n - 1) + n + (n - 1) + \dots + 2 + 1$.

This information is illustrated in the following table.

Boys	Girls	Slides	Jumps
1	1	2	1
2	2	4	4
3	3	6	9
4	4	8	16
5	5	10	25

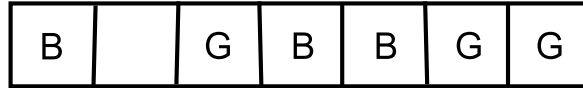
Can we prove that there are always $2n$ slides and n^2 jumps?

Here is a simple argument that there are n^2 jumps: for each of the n boys, he either jumps over or is jumped over by each of the n girls. No other jumps take place. So there must be $n \times n = n^2$ jumps.

Can you prove that the number of slides is $2n$? Furthermore, is it true that every single player slides exactly once?

Diversion 1: Algorithm for Frogs

Let's look at a situation where the players are stuck.



We run into a dead end whenever a move is made that brings two players of the same gender together in the middle of the board. So we must avoid this. How do we ensure that?

In this game, there are always two possible moves (with a couple of exceptions). One of these moves will either bring two players of the same gender together in the middle of the board immediately, or guarantee that this must happen in the next move.

Here is a simple algorithm to solve the problem: “find out which of the two moves does this, and make the other move”. This algorithm always guarantees success.

Here are two extensions to consider.

1. Can you *prove* that this algorithm solves the problem?
2. If you are a computer science teacher, here is a great assignment problem: get your students to write a computer program that plays FROGS, and solves the game for an arbitrary n .

Diversion 2: Mathematical Induction

Quite often when Mathematical Induction is introduced, the examples are highly technical and involve plenty of messy algebraic manipulation. I prefer to introduce Mathematical Induction by using the following example from FROGS:

If you have 1 boy and n girls, how many moves are required to solve the problem?

The formula is quickly seen to be $2n + 1$, and there is a beautiful visual and contextual way to *see* the induction argument. No messy algebraic manipulation!

This is a much nicer way to illustrate Mathematical Induction than the typical textbook example of

$$1 + 2 + 3 + \dots + n = \frac{n(n + 1)}{2}.$$

Speaking of this formula, check out the next diversion:

Diversion 3: Sums of Series

Earlier, we found that if there are n boys and n girls, then the number of jumps is:

$$1 + 2 + 3 + \dots + (n - 1) + n + (n - 1) + \dots + 3 + 2 + 1$$

What does this formula equal?

Since $1 + 2 + 3 + \dots + n = \frac{n(n + 1)}{2}$, we have

$$\begin{aligned} & 1 + 2 + 3 + \dots + (n - 1) + n + (n - 1) + \dots + 3 + 2 + 1 \\ = & (1 + 2 + 3 + \dots + n) + (1 + 2 + 3 + \dots + (n - 1)) \\ = & \frac{n(n + 1)}{2} + \frac{n(n - 1)}{2} \\ = & \frac{n^2 + n}{2} + \frac{n^2 - n}{2} \\ = & n^2. \end{aligned}$$

This provides a great context to discuss the significance and history of the formula for $1 + 2 + 3 + \dots + n$.

There are at least ten ways to develop the formula, here we provide just three:

- (a) Gauss' Pairing Method.
- (b) Handshakes Proof: if there are $(n + 1)$ people at a party, then the number of handshakes equals $1 + 2 + 3 + \dots + n$. But this number also equals $\frac{n(n + 1)}{2}$ because each of the $(n + 1)$ people shook n hands, and we divide the total by 2 since each handshake is counted twice.
- (c) Another Handshakes Proof: if there are $(n + 1)$ people at a party, then the number of handshakes equals $1 + 2 + 3 + \dots + n$. But this number also equals $\binom{n + 1}{2} = \frac{n(n + 1)}{2}$, since each pair of individuals shakes hands.

A Formal Proof:

Here we provide a proof that $n^2 + 2n$ moves are required if there are n boys and n girls.

Consider any of the n boys. How many positions must each of the boys travel to complete the game? This number is seen to be $n + 1$. Similarly, each of the n girls must move $n + 1$ positions, but in the opposite direction. Thus, each of the $2n$ players must travel a total of $n + 1$ positions, accounting for a total of $2n(n + 1)$ moves. However, for each of the n^2 jumps that occur, a player moves over two positions. And this adds an extra n^2 to our count. So the total number of moves required must be $2n(n + 1) - n^2 = n^2 + 2n$, and we are done.

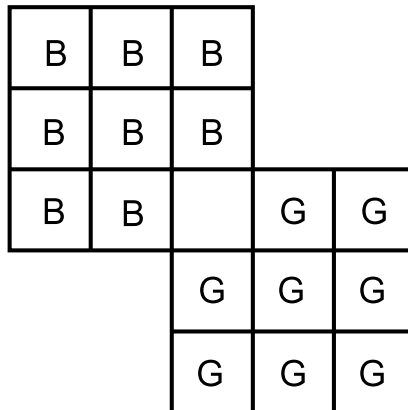
Future Investigations

Here we describe other investigations you can do with your students. There is so much more mathematics to be discovered in this problem!

1. What happens if we have m boys and n girls? How many slides and jumps are required in this generalized game? Can you prove it?
2. Earlier we looked at a move sequence, classifying every move as a slide or a jump (e.g. SJSJJSJS). Now look at the move sequence, classifying every move depending on the gender of the player. For example, the move sequence for the $n = 2$ game will be BGGBBGGB (or equivalently, GBBGGBBG). Investigate this, and try to come up with a general formula for the move sequence for an arbitrary n .
3. What happens if we have an extra space between the players? What is the general formula for the number of moves required if we have m boys, n girls, and p spaces in between?



4. Consider this two-dimensional variant, where boys are only allowed to move right and down, and girls are only allowed to move left and up. Is it possible to solve this problem? If so, how many moves are required? Come up with some other two-dimensional variants, and solve them.



5. Do you remember the game Hi-Q? Use the ideas from FROGS and see if you can solve the Hi-Q game. You can play the game online at:

http://www.lilgames.com/marble_solitaire.shtml