# HOW MANY RECTANGLES?

OAME Leadership Conference

Keynote Presentation

February $24^{\text{th}}$, 2005

Richard Hoshino

# Fun Warmup Problems

As a little warmup, fill out the missing letters.

See how many you can get!

For example, 18 **H** on a **G C** is “18 Holes on a Golf Course”.

24 **H** in a **D**

7 **D** in a **W**

90 **D** in a **R A**

29 **D** in **F** of a **L Y**

9 **P** in the **S S**

64 **S** on a **C B**

88 **K** on a **P**

200 **D** for **P G** in **M**

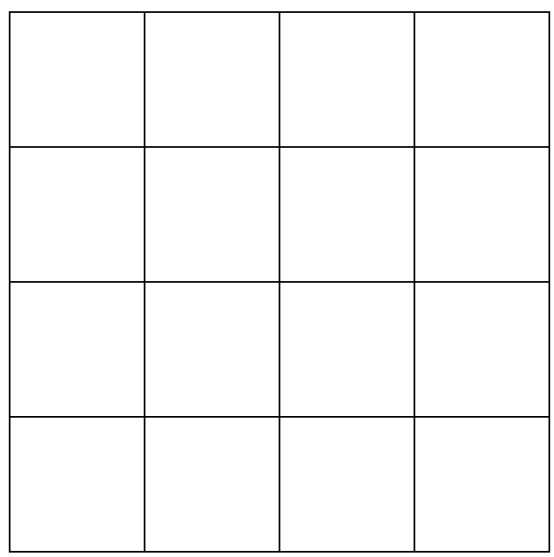
23 **P** of **C** in the **H B**

2 **S** of **R** in a **P** of **K R B**

# The Rectangles Problem

We will spend the next 60 minutes unpacking the mathematics involved in *ONE* exploratory problem. Here is the problem.

**Determine the number of rectangles in the following diagram.**



(Remember that squares count as rectangles too!)

# The First Solution

For each possible case, count the number of rectangles by hand. Here is the final table.

| | |
|---|---|
| 1 by 1 | **16** |
| 1 by 2 | **12** |
| 1 by 3 | **8** |
| 1 by 4 | **4** |
| 2 by 1 | **12** |
| 2 by 2 | **9** |
| 2 by 3 | **6** |
| 2 by 4 | **3** |
| 3 by 1 | **8** |
| 3 by 2 | **6** |
| 3 by 3 | **4** |
| 3 by 4 | **2** |
| 4 by 1 | **4** |
| 4 by 2 | **3** |
| 4 by 3 | **2** |
| 4 by 4 | **1** |

Adding, we find that there are **100** rectangles in the diagram.

# The First Solution (Refined)

This is the same solution as the first, except we recognize that in each case, the number of rectangles can be expressed as a product.

| | |
|---|---|
| 4 by 4 | $\mathbf{1} \times \mathbf{1}$ |
| 4 by 3 | $\mathbf{1} \times \mathbf{2}$ |
| 4 by 2 | $\mathbf{1} \times \mathbf{3}$ |
| 4 by 1 | $\mathbf{1} \times \mathbf{4}$ |
| 3 by 4 | $\mathbf{2} \times \mathbf{1}$ |
| 3 by 3 | $\mathbf{2} \times \mathbf{2}$ |
| 3 by 2 | $\mathbf{2} \times \mathbf{3}$ |
| 3 by 1 | $\mathbf{2} \times \mathbf{4}$ |
| 2 by 4 | $\mathbf{3} \times \mathbf{1}$ |
| 2 by 3 | $\mathbf{3} \times \mathbf{2}$ |
| 2 by 2 | $\mathbf{3} \times \mathbf{3}$ |
| 2 by 1 | $\mathbf{3} \times \mathbf{4}$ |
| 1 by 4 | $\mathbf{4} \times \mathbf{1}$ |
| 1 by 3 | $\mathbf{4} \times \mathbf{2}$ |
| 1 by 2 | $\mathbf{4} \times \mathbf{3}$ |
| 1 by 1 | $\mathbf{4} \times \mathbf{4}$ |

The sum of these numbers is

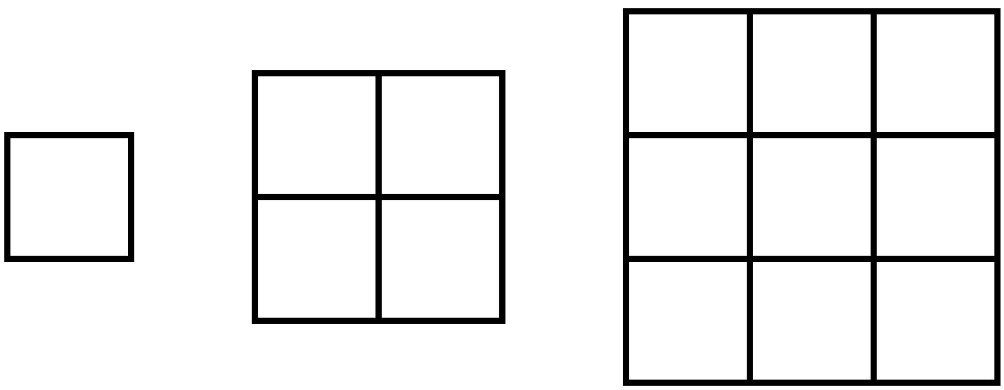
$$\begin{aligned} & 1 \times (1+2+3+4) + 2 \times (1+2+3+4) + \\ & \qquad 3 \times (1+2+3+4) + 4 \times (1+2+3+4) \\ = {} & (1+2+3+4) \times (1+2+3+4) \\ = {} & (1+2+3+4)^2 \\ = {} & \mathbf{100}. \end{aligned}$$

# The Second Solution

A very useful problem-solving strategy is to try *smaller cases* and search for a pattern.

What if we replaced 4 by 4 with a smaller side length $n$?

How many rectangles (of all sizes) would there be in each of these cases?



| | |
|---|---|
| $n = 0$ | **0** rectangles |
| $n = 1$ | **1** rectangle |
| $n = 2$ | **9** rectangles |
| $n = 3$ | **36** rectangles |
| $n = 4$ | ??? rectangles |

In the sequence $0, 1, 9, 36$, is there a nice pattern?

If so, what is it?

# Pattern Recognition

Observe that the numbers 0, 1, 9, and 36 are all perfect squares.

$$\begin{aligned} 0 &= 0^2 \\ 1 &= 1^2 \\ 9 &= 3^2 \\ 36 &= 6^2 \end{aligned}$$

Is there something special about the numbers 0, 1, 3, and 6? Yes! Look at the differences between successive terms.

$$\mathbf{0} \underbrace{\quad}_{1} \mathbf{1} \underbrace{\quad}_{2} \mathbf{3} \underbrace{\quad}_{3} \mathbf{6}$$

In other words,

$$\begin{aligned} 0 &= 0^2 \\ 1 &= (1)^2 \\ 9 &= (1+2)^2 \\ 36 &= (1+2+3)^2 \end{aligned}$$

Continuing this pattern, it appears that the answer to problem for the 4 by 4 case is

$$(1+2+3+4)^2 = \mathbf{100},$$

and this confirms our earlier solution.

# More Pattern Recognition!

We were able to get our answer of **100** in two different ways. Let's look carefully at the sequence of numbers

$$0, 1, 9, 36, 100.$$

Is there anything more we can say about this?

When we look at the differences between successive terms, we get the following:

$$\mathbf{0} \underbrace{\quad}_{1} \mathbf{1} \underbrace{\quad}_{8} \mathbf{9} \underbrace{\quad}_{27} \mathbf{36} \underbrace{\quad}_{64} \mathbf{100}$$

These differences are all perfect cubes!

Earlier we showed that the number of rectangles on a 4 by 4 checkerboard is **100**, which can be written as $(1 + 2 + 3 + 4)^2$.

But 100 can also be expressed as $1^3 + 2^3 + 3^3 + 4^3$.

In other words, we have

$$1^3 + 2^3 + 3^3 + 4^3 = (1 + 2 + 3 + 4)^2.$$

Surely that's a coincidence, eh?

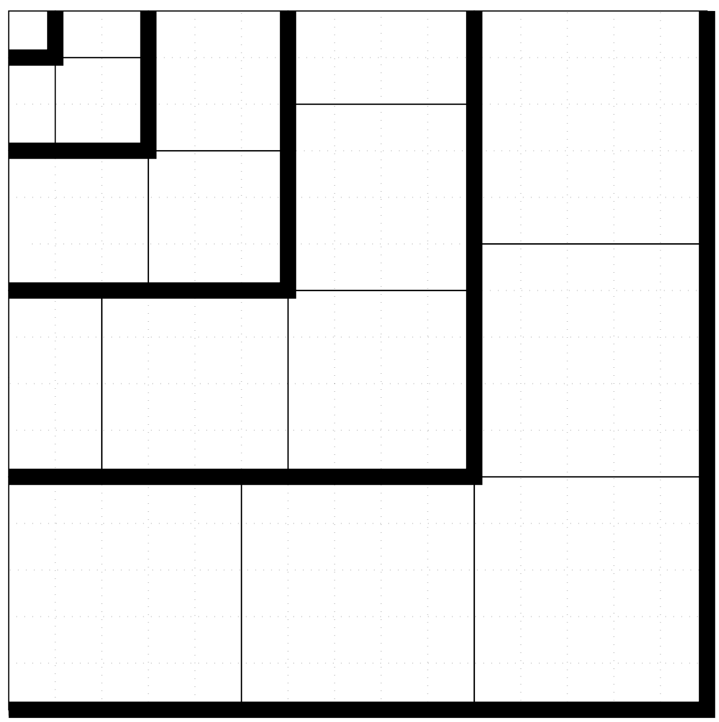
# The Sum of Cubes Identity

Here is one of the most beautiful identities in mathematics:

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

In other words, the sum of the first $n$ cubes is always equal to the square of the sum of the first $n$ positive integers.

We prove this geometrically. Let's do the $n = 5$ case together.



Counting the number of unit squares in this diagram in two different ways, try to explain why

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = (1 + 2 + 3 + 4 + 5)^2.$$

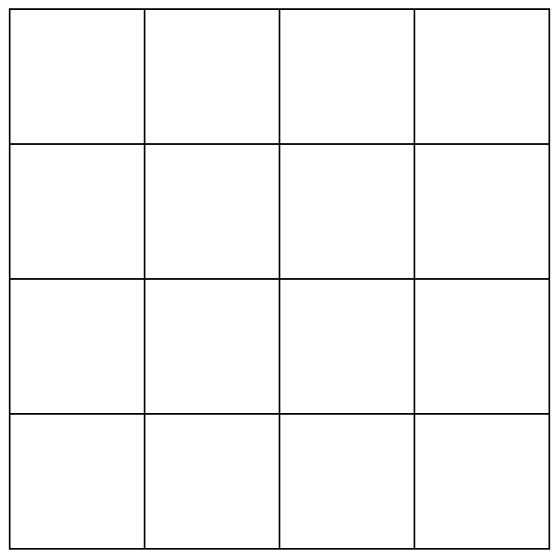
# Making Artists of Our Students

Here is a quote from Peter Taylor (Queen's University).

*What we offer our students is a mess of technical details, with no big picture to see or construct. To make artists of them, we have to start with the whole canvas, teeming with ideas, problems, possibilities, and teach them to say, "don't ply me with details, I don't want all your details, let ME ask the questions". And then they imagine for a while, and then they start a few preliminary sketches, and then they get hungry for facts, theorems, and techniques, not your results, but theirs. And the picture at the end is also not yours but theirs.*

# A Final Solution

We've already found two solutions to the Rectangles problem. Let's find one more.



Choose any two of the five vertical lines.

Choose any two of the five horizontal lines.

What do we get?

# A Final Solution (Continued)

Here is the critical insight:

*Any choice of two vertical lines and two horizontal lines determines a unique rectangle in our checkerboard. Moreover, every rectangle is accounted for by this method.*

So the number of rectangles in our 4 by 4 checkerboard is equivalent to the following problem:

**Determine the number of ways we can select two vertical lines (from a group of five) and two horizontal lines (from a group of five).**

There are 10 ways to select the two horizontal lines, and 10 ways to select the two vertical lines.

Therefore, the answer to the problem is $10 \times 10 = \mathbf{100}$, which is the number of rectangles in our checkerboard.

# A New Context

Consider the following problem:

**I have four free tickets to a Michael Bolton concert, and I want to give them away to two male friends and two female friends, chosen from the following list:**

**Andrea, Breanna, Carolyn, Donna, Ellie,**

**Fabio, Gord, Howard, Ian, Jalen.**

**How many different ways can I give out the tickets?**

This problem is *identical* to our 4 by 4 Rectangles Problem!

# Final Remarks

Thank you for giving me the privilege of sharing my love of mathematics with you. I'd like to conclude with two points.

1. A beautiful mathematical problem will be elegant rather than technical, and highlight powerful mathematical ideas. Consider the mathematics you teach your students. Is the mathematics beautiful and elegant, or just technical?

2. Let's encourage more investigative problem-solving in our classrooms. The best learning often occurs from our failures, and not from our successes. To quote Alfie Kohn, *teachers who want to encourage intellectual growth give students time to be confused and create a climate where it is perfectly acceptable to fall on your face.* Let us focus more on the process, and not just the product.

**My Contact Information:**

Richard Hoshino

hoshino @ mscs.dal.ca

**Website of the Canadian Mathematical Society**

http://www.cms.math.ca

# Dare to Risk!

To laugh is to risk appearing the fool.

To weep is to risk appearing sentimental.

To reach for another is to risk involvement.

To expose your ideas, your dreams, before a crowd is to risk their loss.

To love is to risk not being loved in return.

To live is to risk dying.

To believe is to risk failure.

But risks MUST be taken, because the greatest hazard in life is to risk nothing.

The people who risk nothing, do nothing, have nothing, are nothing.

They may avoid suffering and sorrow, but they cannot learn, feel, change, grow, love, live.

Chained by their attitudes, they are slaves; they have forfeited their freedom.

Only a person who risks is free.