The Million Dollar Hat Problem

Sacred Heart Math Day

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Who Wants to Win a Million Dollars?

Three players enter a room and a red or blue hat is placed on each person's head. The colour of each hat is determined by a coin toss, with the outcome of one coin toss having no effect on the others. Each person can see the other players' hats but not her own.

No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, the players must simultaneously guess the colour of their own hats or pass. The group shares a one million dollar prize if *at least one player guesses correctly and no players guess incorrectly.*

I'll invite three of you to come up and play the game. In my hand is a cheque for $\$1,000,000$. You have *one* chance to play the game. If you win, the cheque is yours. Good luck!

The Fifty-Fifty Strategy:

Here's a simple strategy with a fifty percent chance of winning. Pick one person to be the "captain". Say Katie is our captain. She guesses red. The other two stay silent. If Katie's hat is red, the team wins, otherwise the team loses.

The Worst Strategy:

Let all three guess randomly. Then the probability of winning the game is a meagre $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. That's bad.

The Best Strategy:

Here is a strategy that gives you a 75 percent chance of winning the game. The eight possibilities for the hats are: RRR, RRB, RBR, RBB, BRR, BRB, BBR, and BBB. (By this notation, RBR means that players 1 and 3 have a red hat, and player 2 has a blue hat). Notice that in exactly six cases, there are two hats of one colour and one hat of the opposite colour.

The group can win every time this happens by using the following strategy: once the game starts, each player looks at the other two players' hats. If the two hats are different colours, she passes. If they are the same colour, the player guesses her own hat is the *opposite* colour. This way, every time the hat colours are distributed 2 and 1, one player will guess correctly and the others will pass, and the group will win the game.

When all the hats are the same colour, however, all three players will guess incorrectly and the group will lose.

Now consider the Hat Problem when we have n players, where n is some positive integer.

For $n = 1$, there is a strategy for which the group wins with probability $\frac{1}{2} = 1 - \frac{1}{2}$.

For $n = 3$, there is a strategy for which the group wins with probability $\frac{3}{4} = 1 - \frac{1}{4}$.

For $n = 7$, there is a strategy for which the group wins with probability $\frac{7}{8} = 1 - \frac{1}{8}$.

For $n = 15$, there is a strategy for which the group wins with probability $\frac{15}{16} = 1 - \frac{1}{16}$.

And this pattern continues indefinitely.

In other words, as we increase the number of players, the probability that the group wins the million dollars gets closer and closer to 100%. Provided they know the optimal strategy, they should win the game virtually every time!

The key idea is to ensure that most of the time no one is wrong and occasionally make sure everyone is wrong at the same time.

For example, we saw how in the case $n = 3$, we had exactly one person being right 6 times out of 8, and we had all three people being wrong 2 times out of 8.

For the rest of the talk, we will discuss the optimal strategy for 7 players. In order to do this, we need to introduce ideas from an exciting new area of mathematics called **Coding Theory**.

We will also discuss applications of the Hat Problem to situations we encounter in everyday life, such as purchasing groceries, purchasing an item on the internet, validating a Social Security Card, and checking out books at the library.

To illustrate, let me show you a magic trick.

Does anyone have a book?

The Optimal Strategy for 7 Players

Number the players from 1 to 7. Each of these numbers can be written as a 3-digit *binary number*, so associate each person with a binary number.

Each player will calculate her own **special number**, which will be the **nim sum** of all the players she sees whose hats are red.

For example, say I am player 3.

And I notice that players 1, 2, 5, and 7 are wearing red hats.

Then, my special number is:

$$
\begin{array}{rc}
 & 001 \\
+ & 010 \\
+ & 101 \\
+ & 111 \\
 & - - - - - \\
= & 001
\end{array}
$$

Here is the optimal strategy:

Each player calculates her own special number. If this special number is 000, she guesses that her own hat is red. If this special number is her own number (e.g. she is player 5 and her special number is 101), then she guesses that her own hat is blue. If the special number is anything else, she passes.

Example 1: exactly one person guesses correctly

Let's suppose that the seven people have the following hats:

R,B,B,R,R,B,R

So players 1, 4, 5, and 7 have red hats and everyone else is wearing a blue hat.

The nim sum of 001, 100, 101, and 111 is **111**. We'll call this the **Total Nim Sum** of the seven numbers.

The binary number 111 corresponds to the number 7.

What will be the special number of Player 7?

Let's consider the other people wearing red hats. What will their special numbers be? Let's take an example, say Player 4.

Finally, let's consider the people wearing blue hats. What will their special numbers be? Let's take an example, say Player 6.

Thus, **6 people will pass, and 1 will guess correctly**. The team wins.

Example 2: exactly one person guesses correctly

Let's suppose that the seven people have the following hats:

B,B,B,R,R,B,R

So players 4, 5, and 7 have red hats and everyone else is wearing a blue hat.

The Total Nim Sum is $100 \oplus 101 \oplus 111 = 110$.

The binary number 110 corresponds to the number 6.

What will be the special number of Player 6?

Let's consider the people wearing red hats. What will their special numbers be? Let's take an example, say Player 7.

Finally, let's consider the other people wearing blue hats. What will their special numbers be? Let's take an example, say Player 1.

Thus, **6 people will pass, and 1 will guess correctly**. The team wins.

Example 3: all seven people guess incorrectly

Let's suppose that the seven people have the following hats:

R,B,R,R,B,R,B

So players 1, 3, 4, and 6 have red hats and everyone else is wearing a blue hat.

The Total Nim Sum is $001 \oplus 011 \oplus 100 \oplus 110 = 000$.

Let's consider the people wearing blue hats. What will their special number be? Let's take an example, say Player 5.

Let's consider the people wearing red hats. What will their special number be? Let's take an example, say Player 6.

So if the players employ the strategy, what happens?

Every person wearing a blue hat will guess that her hat is red.

Every person wearing a red hat will guess that her hat is blue.

So **all seven people guess incorrectly**.

So to summarize, this is what happens:

Take all the people with red hats. If the Total Nim Sum of all the "Red-Hat" people adds up to 000, then all seven players will guess incorrectly.

If this Total Nim Sum is not 000, then exactly one of the seven players will guess correctly. For example, if the Total Nim Sum is 011, then player 3 will correctly guess her hat. (The rest stay silent).

There are $2^7 = 128$ ways that the seven people can be assigned the hats. Of these 128 possibilities, exactly 16 have a total nim sum of 000. Here are the sixteen possibilities (note: 1 = RED, 0 = BLUE).

0000000	1000011
0001111	1001100
0010110	1010101
0011001	1011010
0100101	1100110
0101010	1101001
0110011	1110000
0111100	1111111

So the team will lose exactly $\frac{16}{128} = \frac{1}{8}$ of the time. In other words, they will win with probability $\frac{7}{8}$.

Some Historical Context

The Hat Problem was created by Todd Ebert, who introduced it in his Ph.D. thesis at the University of California in 1998.

It turns out that the n player hat problem has "deep and unexpected connections to coding theory, an active area of mathematical research with broad applications in telecommunications and computer science." (New York Times Article, April 2001).

In January 2001, Elwyn Berlekamp, a professor at Berkeley, noticed the connection between the hat problem and Hamming codes. Hamming codes are mathematical structures that are used in error-correcting codes and covering codes.

Error-correcting codes are used in everything from cell phones to compact discs. Covering codes can be used to compress data so they take up less space in a computer's memory.

When you take the Hat Problem apart and look at its core, you will see that what you need are exactly Hamming codes.

There are 2^7 ways of putting hats on the seven people. Each possibility corresponds to a unique binary string of length 7.

For example, if players $1, 3, 4, 6$ have red hats and everyone else has a blue hat, then the corresponding binary string is 1011010.

As we discussed earlier, there are exactly $2^4 = 16$ cases when the team will *lose* the hat problem. These sixteen strings are called the Hamming codewords for the case $n = 7$.

0000000	1000011
0001111	1001100
0010110	1010101
0011001	1011010
0100101	1100110
0101010	1101001
0110011	1110000
0111100	1111111

These 16 strings have the property that any pair of them differ in at least three positions. What can we do with this?

Transmission of a Message Over an Unreliable Channel

Since there are 2^4 of these codewords, we can associate each 7-bit codeword with a unique 4-bit string.

$$
\begin{array}{ll}
\mathbf{A}:0000 \to \mathbf{0000000} & \mathbf{I}:1000 \to \mathbf{1000011} \\
\mathbf{B}:0001 \to \mathbf{0001111} & \mathbf{J}:1001 \to \mathbf{1001100} \\
\mathbf{C}:0010 \to \mathbf{0010110} & \mathbf{K}:1010 \to \mathbf{1010101} \\
\mathbf{D}:0011 \to \mathbf{0011001} & \mathbf{L}:1011 \to \mathbf{1011010} \\
\mathbf{E}:0100 \to \mathbf{0100101} & \mathbf{M}:1100 \to \mathbf{1100110} \\
\mathbf{F}:0101 \to \mathbf{0101010} & \mathbf{N}:1101 \to \mathbf{1101001} \\
\mathbf{G}:0110 \to \mathbf{0110011} & \mathbf{O}:1110 \to \mathbf{1110000} \\
\mathbf{H}:0111 \to \mathbf{0111100} & \mathbf{P}:1111 \to \mathbf{1111111}
\end{array}
$$

So we can transmit binary information over an unreliable channel as follows:

1. Separate the message into blocks of 4.

2. Convert each block to the unique 7-bit Hamming code, and send it through the channel.

3. This unreliable channel may flip a 0 to a 1 accidentally, or vice-versa. If exactly one such error occurs, then the Total Nim Sum of the red hat numbers will not be 000.

   However, this is exactly the hat problem: if the Total Nim Sum is not 000, we can fix it. Say the Total Nim Sum is 110. Then we take the sixth bit, and flip it. Now the Total Nim Sum is 000.

Applications

Each of these 2^4 Hamming codewords of length 7 can correctly detect one error, find where the error occurred, and flip that bit to reproduce the correct string.

However, if there was more than one error, then there is no way for the receiver to realize that. But it is highly unlikely that two mistakes will occur in the transmission of seven bits.

This process makes you lose some efficiency, since you are sending seven bit strings instead of four bit strings. However, in return, you get *reliability*.

And that is the idea behind codewords. We add extra bits to ensure that we have scanned in the data correctly, and if we haven't, we can correct them.

Here are some everyday applications that involve error detecting and error correcting codes.

Purchasing Books: **ISBN** (International Standard Book Number)

Using a Credit Card: **IBM Check** (for VISA and Mastercard)

Sending Money: **EFT** (Electronic Funds Transfer)

Buying Groceries: **UPC** (Universal Product Code)