# COMBINATORICS PROBLEMS

Math Teachers Workshop at Waterloo

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# Sesquipedalian Expressions

Sesquipedalian Expressions use unnecessarily big words. Your task is to translate each of these expressions into "common English", i.e., cliches and phrases that you know.

1. It is fruitless to attempt to indoctrinate a superannuated canine with innovative maneuvers.
2. Pulchritude possesses solely cutaneous profundity.
3. It is fruitless to become lachrymose over precipitately departed lactose fluid.
4. Freedom from encrustation of grime is contiguous to divinity.
5. Individuals who make their abode in vitreous edifices would be advised to refrain from catapulting petrous projectiles.
6. Where there are visible vapors having their province in ignited carbonaceous material there is conflagration.
7. Exclusive dedication to necessary chores without interludes of hedonistic diversion renders Jack a hebephrenic fellow.
8. Missiles of ligneous or petrous consistency have the potential of fracturing my osseous structure but appellations will eternally be benign.

# The Handshakes Problem

**There were 10 people who attended a party.**

**Everybody shook hands with everyone else.**

**How many handshakes took place?**

Come up with at least *THREE* solutions to this problem.

(Work in groups of 1-3 people)

*"The important question is, how many hands have I shaked?"*

– George W. Bush, campaigning in New Hampshire.

# Representations of a Math Concept

According to Lesh, Post, and Behr (1987), there are five different representations of a mathematical concept.

1. Symbolic
2. Contextual
3. Linguistic
4. Pictorial
5. Concrete

Strengthening the ability to move between and among these five representations improves the growth of students' understanding of mathematical concepts.

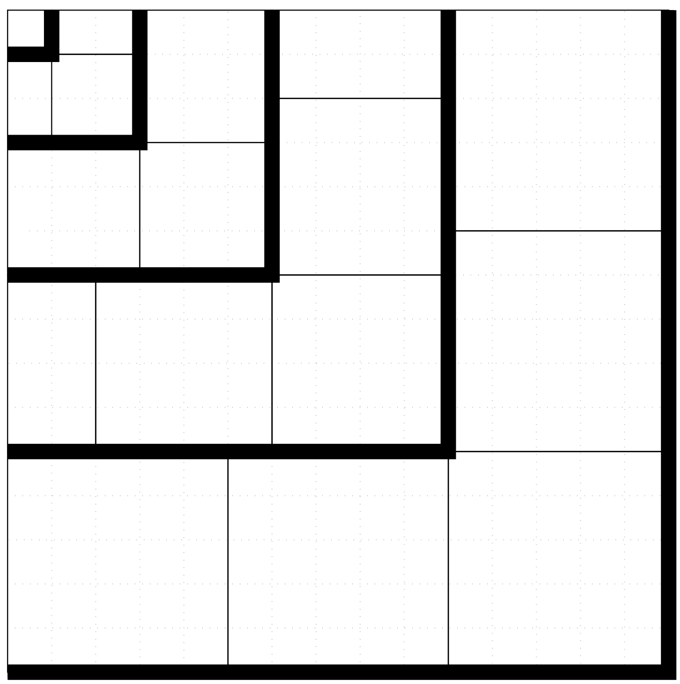
# The Sum of Cubes Identity

Here is one of the most beautiful identities in mathematics:

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

In other words, the sum of the first $n$ cubes is always equal to the square of the sum of the first $n$ positive integers.

We prove this geometrically. Let's do the $n = 5$ case together.



Counting the number of unit squares in this diagram in two different ways, try to explain why

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = (1 + 2 + 3 + 4 + 5)^2.$$

# Mr. and Mrs. Smith

Mr. and Mrs. Smith were at a party with three other married couples. Since some of the guests were not acquainted with one another, various handshakes took place. No one shook hands with his or her spouse, and of course, no one shook their own hand! After all of the introductions had been made, Mrs. Smith asked the other seven people how many hands each shook. Surprisingly, they all gave different answers. How many hands did Mr. Smith shake?

# SIM

Here are the rules of the game *SIM*, played between two players, Amy and Bob. We start with six points on the board.

Amy receives a red piece of chalk, and Bob receives a blue one. On their turn, the player draws an edge between any pair of points. The first person to create a *monochromatic triangle* loses.

Amy moves first. Who wins the game?

More interestingly, *must* there be a winner?

# Investigations

In groups of 3-4, you will work on one of the following two investigations.

**Problem 1**: An Introduction to Ramsey Theory

**Problem 2**: Curling Bonspiels and Score Sequences

In your group, spend an hour working on the investigation. We will have a short "debriefing" session to complete this afternoon's session.

For more information on "Combinatorial Explorations" and other resources that might interest you, visit the website of the Canadian Mathematical Society.

http://www.cms.math.ca

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# Investigation 1

## An Introduction to Ramsey Theory

This is a guided investigation that will introduce the fundamentals of Ramsey Theory, a branch of discrete mathematics that is an area of active research today. Consider this problem.

**Determine the smallest positive integer $n$ for which the following statement is true: if $n$ people attend a party, then there must be three mutual acquaintances, or three mutual strangers.**

From our earlier discussion, we showed that the answer is $n = 6$. We say that the **Ramsey number** $R(3,3)$ equals 6.

We define the Ramsey number $R(x,y)$ to be the smallest integer $n$ for which the following statement is true: if $n$ people attend a party, then there *must* either be a group of $x$ mutual acquaintances, or a group of $y$ mutual strangers. In our definition, we require $x \geq 2$ and $y \geq 2$.

1. Can you find and prove a formula for $R(2,n)$? Hint: the formula is very easy!

2. Prove that in any group of 10 people, there must be a group of 3 mutual acquaintances, or a group of 4 mutual strangers. In other words, you want to show that if you take 10 points, and colour each edge either red or blue, then there must either be three vertices all connected by red edges (i.e., an all-red triangle), or four vertices all connected by blue edges (i.e., an all-blue four-set).

   Your proof will be along the same lines as our proof that $R(3,3) = 6$. To start off, consider point $P$, and look at the nine edges incident with $P$. Each of these nine edges is either red or blue. Separate your analysis into these two cases:

   (a) Case 1: at least four of these nine edges are red.

   (b) Case 2: at least six of these nine edges are blue.

   Can you take it home from there? (Hint: for the second case, you will want to use the fact that $R(3,3) = 6$).

3. Now let's go one step further, and show that in any group of 9 people, there must be a group of 3 mutual acquaintances, or a group of 4 mutual strangers. We'll employ a Proof by Contradiction. Suppose that we can colour the edges of our graph on 9 points so that we do not create an all-red triangle, or an all-blue four-set. Let's show that this is impossible.

Consider point $P$, and look at the eight edges incident with $P$. Each of these eight edges is either red or blue. From the previous question, we've solved the problem if $P$ is adjacent to at least four red edges or $P$ is adjacent to at least six blue edges. So we may ignore these cases.

So what is the only remaining case? How many blue edges and red edges must we have from $P$? Convince yourself that the only remaining case occurs when $P$ has three red edges and five blue edges. Now by symmetry, explain why we can assume that *every* vertex must be incident with three red edges and five blue edges.

In other words, the only remaining possibility occurs when our graph consists of 9 vertices, where every vertex is incident with three red edges and five blue edges. Can you show that such a graph cannot exist? This will complete the proof.

4. To complete the problem, arrange a party with 8 people so that we do *not* have three mutual acquaintances, or four mutual strangers. To do this, create a graph on 8 points where the edges are coloured in such a way that there is no all-red triangle or an all-blue four-set. This will prove that $R(3,4) = 9$.

5. Prove that $R(x, y) \leq R(x-1, y) + R(x, y-1)$, for any integers $x, y \geq 2$.

   This is a more general form of the technique you used to prove Question 2. To illustrate, proving this theorem shows that $R(3,4) \leq R(2,4) + R(3,3) = 4 + 6 = 10$, which was what Question 2 sought to establish.

6. Now prove that if $R(x-1, y)$ and $R(x, y-1)$ are both even, then we have $R(x, y) \leq R(x-1, y) + R(x, y-1) - 1$, for any integers $x, y \geq 2$.

   Basically, this is the generalized version of Question 3. Proving this theorem shows that $R(3,4) \leq R(2,4) + R(3,3) - 1 = 9$, which was what Question 3 sought to establish.

7. Using Questions 5 and 6, can you guess the correct values of $R(3,5)$ and $R(4,4)$?

Despite extensive research, very little is known about these Ramsey numbers. The only known Ramsey numbers are $R(3,3)$, $R(3,4)$, $R(3,5)$, $R(3,6)$, $R(3,7)$, $R(3,8)$, $R(3,9)$, $R(4,4)$, and $R(4,5)$. Half of these were found using the computer. For many years, it has been known that $R(5,5)$ is somewhere between 43 and 49, but no one has been able to do any better than that!

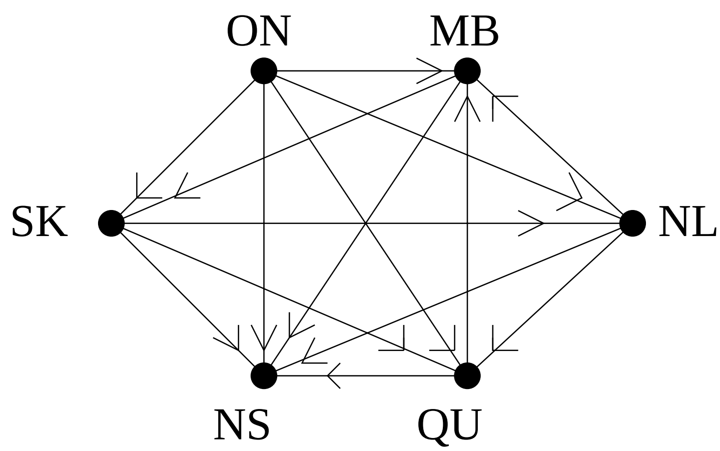
Ramsey theory is a fascinating branch of mathematics that makes important connections to many different areas of pure and applied mathematics. Finding a formula for $R(x, y)$ is one of the most difficult unsolved problems in discrete mathematics, and no one is anywhere close to a solution.

# Investigation 2

## Curling Bonspiels and Score Sequences

Consider a curling bonspiel with 6 teams: Nova Scotia (NS), Quebec (QU), Manitoba (MB), Newfoundland & Labrador (NL), Saskatchewan (SK) and Ontario (ON). Each team plays each other team exactly once in this round-robin event. The winner of each game is awarded 1 point, whereas the loser is awarded 0 points. Curling matches do not end in ties.

We will draw a graph indicating the results of each game in the tournament:




For example, the arrow from Ontario (ON) to Manitoba (MB) means that Ontario lost to Manitoba. From the graph, we can determine the result of each game. Each team's score can be found by counting the number of arrows that point towards that team's vertex. The number of points for each team is indicated in the table:

| Team Name | Score |
|---|---|
| NS | 5 |
| QU | 3 |
| MB | 3 |
| NL | 2 |
| SK | 2 |
| ON | 0 |

Thus, the **score sequence** for this tournament is $(0, 2, 2, 3, 3, 5)$. In a score sequence, we list the numbers in increasing order.

In general, for an $n$-team tournament, the score sequence is $(a_1, a_2, \ldots, a_n)$, where $a_1 \leq a_2 \leq \ldots \leq a_n$.

1. Which of the following are valid score sequences? If the score sequence is valid, construct a tournament that gives that score sequence. If the score sequence is invalid (i.e. impossible), carefully explain why this is the case. *(Hint: only two of the following score sequences is valid!)*

   (a) $(0, 1, 2, 3, 4, 5)$.

   (b) $(0, 0, 3, 3, 4, 5)$.

   (c) $(1, 2, 2, 3, 4, 4)$.

   (d) $(1, 2, 2, 3, 4, 6)$.

   (e) $(0, 1, 1, 4, 4, 5)$.

   (f) $(0, 2, 3, 3, 3, 4)$.

   (g) $(0, 1, 2, 2, 5, 5)$.

2. Let $s(n)$ be the number of possible score sequences in a tournament with $n$ teams. Show that $s(3) = 2$, $s(4) = 4$, $s(5) = 9$, and $s(6) = 22$ by listing out all of the valid score sequences for each of these cases. Now look carefully at these valid score sequences. Do you notice anything interesting?

3. Based on Questions 1 and 2, can you come up with some conditions that would show that a given score sequence is impossible? There is one very important condition based on the *running totals* of the numbers in the score sequence, i.e., consider the sums $a_1$, $a_1 + a_2$, $a_1 + a_2 + a_3$, $a_1 + a_2 + a_3 + a_4$, and so on. What is this condition?

4. Find a necessary and sufficient condition for a score sequence $(a_1, a_2, \ldots, a_k)$ to be valid. Can you prove your claim? For example, how would you show that $(1, 2, 2, 3, 3, 5, 5)$ is a valid score sequence? This is a very very hard problem!! (If you are interested in finding out more about this problem, do a Google search on "Landau's Theorem").

This investigation is just the starting point of a fascinating area of mathematical research callde "Tournament Theory".