Introduction to Binomial Coefficients

We define $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$. Note: $n!$ is called "*n factorial*"

For example, there are $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ permutations of the word *MATH* (i.e., different ways of rearranging these four letters to form a word, such as *AMTH* and *HAMT*). This follows since we have four choices for the first letter of each permutation, three possibilities for the second letter, two for the third letter, and one for the final letter.

When letters repeat, the process is a little more complicated. For example, *BALL* does not have $4!$ permutations, but instead, has $\frac{4!}{2!} = 12$ permutations. We must divide by $2! = 2$ because the two L's are the same, and so we are not creating different words by switching the positions of these L's. A fun word to try is *MISSISSIPPI*. It has four I's, four S's and two P's. Thus, the number of permutations of the word *MISSISSIPPI* is $\dfrac{11!}{4! \cdot 4! \cdot 2!}$.

We define $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

Notice how this is exactly the number of ways to rearrange the letters of a word with k A's and $(n-k)$ B's.

For example, if $n = 5$ and $k = 2$, there are $\frac{5!}{2!3!} = 10$ ways of rearranging the letters of the word with two A's and three B's. The ten possibilities are

$$AABBB, ABABB, ABBAB, ABBBA, BAABB,$$
$$BABAB, BABBA, BBAAB, BBABA, BBBAA$$

This is also the number of ways we can choose k elements from a set of n elements. So with $n = 5$ and $k = 2$, there are exactly $\binom{5}{2} = 10$ ways of choosing two elements from the set $\{1, 2, 3, 4, 5\}$. These ten possibilities are:

$$(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5).$$

What is the correspondence between the "words" $AABBB, ABABB$, etc. and the pairs $(1,2), (1,3)$, etc.?

Problem: Say James is at the point $(0,0)$, and he wants to get to $(4,7)$. He can only move one unit right, or one unit up, on any move. A possible path is up, right, up, right, up, up, up, right, right, up, up.

How many such paths are there? Without doing any work whatsoever, can you convince yourself that the answer must be $\dbinom{11}{4}$?