The Ubiquitous Golden Mean

By Ravi Vakil

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Much of mathematics consists, broadly speaking, of finding and explaining unlikely patterns. If something is true in mathematics, there is a good reason for it to be true. If something is true that *seems* to have no right to be true, there is usually a *really* good reason for it to be true. Surprising facts tend to have deep and far-reaching causes: witness how Fermat's Last Theorem, by defying mankind's best minds for centuries, gave rise to much of the modern theory of numbers.

The golden mean τ, a number fascinating even to the ancients, has a habit of turning up in the oddest of places. From the Parthenon to da Vinci to sunflowers to strange number patterns, the golden mean appears repeatedly, often along with Fibonacci numbers, and the number 5. We'll take a leisurely stroll through some of these coincidences, doing some puzzles, resolving some paradoxes, and playing with numbers.

First of all, I'd like to thank the organizers at Wheaton College for inviting me to speak. I'm enjoying the visit, and had a great lunch with some of the students here.

I'd like to talk to you today about what mathematics is all about. It's often very different from the sort of things you see in class.

Mathematics is all about patterns. Observing them, explaining them, proving them, and extending them: finding beauty and order in a seemingly chaotic universe. What's amazing about mathematics, and indeed most other fields of study, is the fact that these fields can exist at all — that the complicated world actually follows simpler patterns, and when you understand them, you find patterns among the patterns, and so on. If this weren't the case, mathematics would have shut down years ago: we would have found all straightforward patterns in the world, and we'd be done.

Mathematics is at its most fun, and most deep, when you find links between seemingly unconnected ideas that seem too good to be true, that seem to have no right to be true. Then you know you've stumbled on some deeper mystery, and then it's a matter of ferreting out clues.

I'll show you some patterns involving famous mathematical numbers, to make you really believe that something deep is going on that I'm not telling you about, that is even *more* beautiful than the things I *am* telling you about. Some of the deep connections I'm not describing are not hard, and some are very hard. If you have some guesses and ideas about these links, I'd be interested in hearing them. They involve algebra, geometry, and occasionally physics and biology, and I'm not convinced that we understand everything that's going on.

The Players.

Fibonacci numbers, defined by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$ $(n \geq 0)$. Here is a table of the first few numbers.

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
0	1	1	2	3	5	8	13	21	34	55	89	144	233	377

These numbers have been with us for a long time. They were attributed to Fibonacci, which was the nickname of Leonardo of Pisa, who lived around 1200 AD. He introduced them in the form of a problem, or puzzle:

How many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on?

Botany. Of course, the rabbit problem is contrived, but it's a convenient way of describing the sequence simply. But Fibonacci numbers come up in a wide variety of contexts. Most amazingly, they come up in nature.

Pineapples, pine cones, sunflowers.

This isn't a coincidence, although it isn't obvious as of yet why there should be a connection.

Miles to km. Here's something that *is* a coincidence. But it's fun.

I'm originally from Canada, where distance is usually measured in kilometers. So it took a bit of a psychological effort to switch into miles. Here's a cute way to convert, that's quite accurate.

Basically, a good approximation for F_n miles is F_{n+1} km, and vice versa: 55 miles is about 89 km, so 55 mph is about 89 kmh. I'll give some clues later on as to why this trick should work. The reason has to do with the fact that 1 mile is about 1.6 km.

The magic number 9899. Here's another example where Fibonacci numbers turn up unexpectedly. The magic number 9899.

$$\frac{1}{9899} = 0.000101020305081321....$$

Try 89 (which happens to be Fibonacci): $\frac{1}{89} = 0.011235...$

Can you think of any more magic numbers? For an example, look at 1/998; what do you notice?

The Golden Mean.

Another number, even older, is the number known as the *golden mean* or *golden section*, $\tau = (1 + \sqrt{5})/2$ ("tau"), which is approximately 1.61803399.

If you plug τ^2 into a calculator, you get 2.61803399, which looks a lot like τ, and in fact you can check that $\tau^2 = \tau + 1$.

The equation $x^2 = x + 1$ has τ as a solution. As it has degree 2, it has another solution, sometimes called σ ("sigma"), τ's poor relation. $\sigma = 1 - \tau = -1/\tau = (1 - \sqrt{5})/2$. It's approximately -0.61803399, and $\sigma^2 = \sigma + 1$.

τ was an important to the ancient greeks. At the time, the various branches of knowledge hadn't yet separated, and math and aesthetics and philosophy were all intertwined. The golden mean was considered one of the most aesthetically beautiful numbers. For example, the *golden rectangle* proportional to a rectangle with sides 1 and τ, was considered the most beautiful rectangle.

Because the golden mean was considered beautiful, it was used in much classical Greek architecture. You can see it all over the Parthenon, and other classical Greek structures. In the renaissance, the golden mean was used in many paintings and sculptures as well.

There are an incredible number of coincidences and links involving the Fibonacci numbers and the golden mean, and I'm going to spend the rest of my time talking about them. In many of these, you'll notice a third player as well: the number 5. For example, 5 has already turned up in the definition of τ.

First link (seemingly tenuous). $\tau^{n+2} = \tau^{n+1} + \tau^n$, $F_{n+2} = F_{n+1} + F_n$.

Approximating τ. τ is an irrational number, which means that you can't express it as a fraction a over b, where a and b are integers. But you can "approximate" it by rationals, and the best approximations are ratios of Fibonacci numbers! $8/5 = 1.6$ is pretty good; $55/34 \cong 1.6176$ is even better. They get better and better the bigger Fibonacci numbers you use.

Related link: a formula for Fibonacci. This fact is true because there is a formula for the Fibonacci numbers

$$F_n = \frac{\tau^n - \sigma^n}{\sqrt{5}}.$$

$\tau^n/\sqrt{5}$. Another thing you'll notice: because σ is really small, σ^n is really really small when n is large, so F_n is very close to $\tau^n/\sqrt{5}$. To show you what this coincidence this implies, I'll make a table of $\tau^n/\sqrt{5}$.

n	$\tau^n/\sqrt{5}$
1	.723607
2	1.170820
3	1.894427
4	3.065248
5	4.959675
6	8.024922
7	12.984597
8	21.009519
9	33.994117
10	55.003636
20	6765.000030

So we have an amazing fact: to find F_n, calculate $\tau^n/\sqrt{5}$, which is some hideous horrible irrational number, and round it to the nearest integer. Better yet, it becomes "easier and easier" to round it, because it gets closer and closer to the integer.

That seems something very special about $\tau^n/\sqrt{5}$, doesn't it? Well, it isn't. Here's another table for τ^n.

n	τ^n
1	1.618034
2	2.618034
3	4.236068
4	6.854101
5	11.090170
6	17.944272
7	29.034442
8	46.978714
9	76.013156
10	122.991869
20	15126.999934

Once again, it's getting closer and closer to an integer. So let's get a new sequence, by taking $\sqrt{5}$ and rounding it to the nearest integer. 2, 3, 4, 7, 11, 18, 29, 47, 76, 123, ... What pattern does it follow? I'll throw away the 2. Fibonacci pattern again! In fact, if you take sum of 2 Fibonacci numbers two terms away in the sequence, you'll get a term in this sequence! For example, 5+13=18. Why should this be?!

Pascal's Triangle. Here's another surprising place where Fibonacci turns up: Pascal's triangle.

Geometry. The golden mean also turns up in surprising contexts, often in geometry. Here are a few Golden mean coincidences. I won't go into the trig involved, but they are related to the facts that $\sin 18 = (\tau - 1)/2$ and $\cos 36 = \tau/2$. If you've dealt with trig, you've undoubtedly memorized the formulas for sine and cos of 30, 60, 90, 45, etc. In fact, other angles like 18 and 36 have nice trig values as well.

Regular decagon (10-sided regular polygon) of side length 1 sits inside a circle. What's its radius? τ.

Pentagram. All the τ's there. Fibonacci.

Icosahedron of side length 1: it's the most complicated platonic solid, and the name platonic solid reminds us that we're going back to the ancient greeks. It has 20 triangular sides. Suppose it has side length 1. Then there are lots of τ's here. For example, this is τ. The volume is something complicated, but it can be written as $a + b\tau$, where a and b are some special numbers.

If you slice off all the corners, you'll be left with a soccer ball, and indeed there are a huge number of golden means running around the soccer ball. It's volume is also $c + d\tau$, and lots of lengths turn out to be powers of golden means.

The Duodecahedron is the second-most complicated platonic solid. It has 12 faces, each of which is a pentagon. Suppose it has side length 1. We've already seen golden means turn up in pentagons, so again the golden mean turns up, and the greeks found it beautiful partially for this reason. Again, its volume is of the form $e + f\tau$ for some integers e and f. And the number 5 turns up as well. Inside the duodecahedron I can show you a cube of side length τ. In fact, there are five such cubes you can find.

Back to the sunflower.

Let me close by returning to botany, and trying to give a beginning of a mathematical explanation of why Fibonacci numbers turn up in sunflowers and pine cones and pineapples, although I may just obfuscate things further. The explanation involves the golden mean.

Consider a circle whose circumference is τ. Start at any point on the circle and take some number of consecutive steps of arc length one in the clockwise direction. If you like thinking in terms of angles, at each step you should turn $360/\tau$ degrees clockwise, which is approximately 222.5 degrees. Label the points you step on in the order you encounter them, labelling your starting position 0, your first step 1, your second step 2, and so on.

When you stop, the difference in any two adjacent numbers is a Fibonacci number.

Conclusion.

In conclusion, I hope I've convinced you that by starting with just a few simple concepts (the Fibonacci numbers and the golden mean), and playing around, you can find an intricate family of connections crying out to be explained. The hunt for these sorts of connections, and the explication thereof, are what mathematics is really all about. These patterns in nature are staring us in the face, and it's only human to want to understand why they are true. Thank you.