Solution for Weekly Proof 1

Prove that a 10 by 10 board cannot be covered with 25 L-shaped quadrominoes.

Assume that we *can* place 25 L-shaped quadrominoes so that every square on the board is covered. Since $25 \times 4 = 100$, notice that every square is covered exactly once, and there is no overlap. Consider such a covering of the board. We will prove that the sum of the numbers of the one hundred squares is *odd*.

Pick a L-shaped quadromino on the board, and add up the numbers of the four squares that the piece covers. We shall show that this sum is odd. There are many ways to do this, such as exhausting all sixteen cases, but here is a clever approach. Every quadromino, regardless of the orientation, can be split into two dominoes: a horizontal domino (with length 2 and width 1) and a vertical domino (with length 1 and width 2). Each horizontal domino sums to an *odd* number, because the two squares that this domino covers has opposite parity. Each vertical domino sums to an *even* number, because the two squares that this domino covers has the same parity (i.e., either 0, 0 or 1, 1.) Thus, the sum of the four squares must be odd, since it is the sum of an odd number and an even number.

Call this sum S_1. Repeat this with all the other quadrominoes, and we get an odd sum each time. So $S_2, S_3, \ldots, S_{25}$ are all odd. Now, the sum of the numbers of the one hundred squares in the board is just $S_1 + S_2 + S_3 + \ldots + S_{25}$, since the twenty-five quadrominoes cover the entire board without any overlap.

But $S_1 + S_2 + S_3 + \ldots + S_{25}$ is odd. Once again, there are many ways to show this. The following is just one of these methods. Since S_i is odd for each i, we can write each S_i as $2T_i + 1$, for some integer T_i. Then $S_1 + S_2 + \ldots + S_{25} = (2T_1 + 1) + (2T_2 + 1) + \ldots + (2T_{25} + 1) = 2(T_1 + T_2 + \ldots + T_{25}) + 25$, which is odd because it is the sum of an even number and an odd number. Therefore, $S_1 + S_2 + \ldots + S_{25}$ is odd.

So the sum of the numbers of the one hundred squares is odd. However, in our numbering of these squares, exactly half of them were numbered 1, and so the sum of the numbers of these one hundred squares must be 50, which is even. Hence, we have a contradiction, and so we conclude that it is impossible to cover a 10 by 10 board with 25 L-shaped quadrominoes.