Solution for Weekly Proof 10

Consider the following true sentence:

In this sentence there are exactly **1** *0's,* **2** *1's,* **3** *2's, and* **2** *3's.*

Notice how there is 1 zero, 2 ones, 3 twos, and 2 threes that appear in the sentence.

Replace the question marks with integers to make the following sentence correct:

In this sentence there are exactly ? 0's, ? 1's, ? 2's, ? 3's, ? 4's, ? 5's, ? 6's, ? 7's, ? 8's, and ? 9's.

Let a_i represent the number of i's there are in the sentence. So each a_i represents a question mark. In the end there will be 20 digits in our sentence and so $a_0+a_1+a_2+\ldots+a_9 = 20$.

Each number appears once in the list so there can only be one number "0". Thus, $a_0 = 1$.

Suppose that $a_9 = 2$. Then another of the a_i's would have to be 9, (say a_j), and so there would be 9 j's in our sentence. That means that eight of the terms in $\{a_0, a_1, a_2, \ldots, a_9\}$ would have to be j, and since $a_0 = 1$ and $a_9 = 2$, we require $a_1 = a_2 = \ldots = a_8 = j$. But then the sum of these numbers would be $3 + 8j$ which cannot equal 20. So a_9 cannot be 2, and it is easy to see that a_9 cannot exceed 2, and so a_9 must be 1. Similarly, one can show that a_8 must be 1.

Try $a_7 = 2$. Then $a_j = 7$ for some j. It is not too difficult to see that if $j > 2$, then when we fill out our table, the sum of the a_i's will exceed 20 (by quite a bit). So we want $a_1 = 7$. Since $a_7 = 2$, we must have $a_2 \geq 2$, but if $a_2 = 2$, then we will have three 2's in our sentence, which will contradict the fact that $a_2 = 2$. So we want $a_2 \geq 3$. Trying $a_2 = 3$, we quickly get $a_3 = 2$ and $a_4 = a_5 = a_6 = a_8 = a_9 = 1$, and checking, we see that these ten numbers add up to 20 and they satisfy the sentence.

So we have $a_0 = 1, a_1 = 7, a_2 = 3, a_3 = 2, a_4 = 1, a_5 = 1, a_6 = 1, a_7 = 2, a_8 = 1, a_9 = 1$, and so our sentence is:

In this sentence there are exactly **1** *0,* **7** *1's,* **3** *2's,* **2** *3's,* **1** *4,* **1** *5,* **1** *6,* **2** *7's,* **1** *8, and* **1** *9.*