

## Solution for Weekly Proof 4

Let  $n$  be an even perfect number. Thus,  $n$  must be of the form  $2^{k-1}M$ , for some integers  $k$  and  $M$ , where  $k \geq 2$ . We'll prove that  $M$  must be a prime number of the form  $2^k - 1$ .

Let  $S(M)$  be the sum of the divisors of  $M$ , including  $M$  itself.

Since  $n$  is a perfect number, the sum of the proper divisors of  $n$  is equal to  $n$ . In other words, the sum of all the divisors of  $n$  (i.e. the proper divisors of  $n$  added to the original number  $n$ ) is equal to  $2n$ . Hence,  $S(n) = 2n$ .

Now,  $S(n) = S(2^{k-1}M)$ . Let the divisors of  $M$  be  $d_1, d_2, \dots, d_r$ . So the divisors of  $n$  will be of the form  $2^i d_j$ , for some  $i$  between 0 and  $k-1$  and some  $j$  between 1 and  $r$ . These are all the possible divisors. Thus, the sum of these divisors can be calculated as follows:

$$\begin{aligned} S(2^{k-1}M) &= (2^0 + 2^1 + 2^2 + \dots + 2^{k-1})(d_1 + d_2 + \dots + d_r) \\ &= (2^k - 1)(d_1 + d_2 + \dots + d_r) \\ &= (2^k - 1)S(M) \end{aligned}$$

Hence, we have  $2n = S(n) = S(2^{k-1}M) = (2^k - 1)S(M)$ . But  $n = 2^{k-1}M$ , so we have  $2^k M = (2^k - 1)S(M)$ . Since  $\gcd(2^k, 2^k - 1) = 1$ ,  $2^k - 1$  must divide  $M$ . Thus,  $M = (2^k - 1)T$ , for some integer  $T$ .

Substituting this back into our formula, we have  $2^k(2^k - 1)T = (2^k - 1)S(M)$ , so  $S(M) = 2^k T$ .

But  $M$  has divisors  $T$  and  $M$ , and hence  $S(M)$  is *at least*  $T + M = T + (2^k - 1)T = 2^k T = S(M)$ . So the only way we could have equality is if  $T$  and  $M$  are the only divisors of  $M$ . In other words, we must have  $M$  being a prime, and  $T$  being 1.

Therefore, if  $n = 2^{k-1}M$  is perfect, then we must have  $M = (2^k - 1)T = 2^k - 1$ , where  $2^k - 1$  is prime. And this was Euler's proof that every even perfect number must be of the form  $2^{k-1}(2^k - 1)$ , where  $M_k = 2^k - 1$  is a Mersenne prime.