Math 4220/5220 - PDE's

Mid-Term Test Solutions

1. (a) Solve the equation $u_x + 2u_y + u = 1$, $u(0,y) = y^2$.
Let

$$\begin{aligned}\eta &= y - 2x\,,\\ \xi &= x\,.\end{aligned}$$

Then

$$\begin{aligned}u_x &= u_\eta(-2) + u_\xi\,,\\ u_y &= u_\eta\,.\end{aligned}$$

Subbing in gives

$$\begin{aligned}-2u_\eta + u_\xi + 2u_\eta + u &= 1\,,\\ u_\xi + u &= 1\,,\\ (e^\xi u)_\xi &= e^\xi\,,\\ e^\xi u &= e^\xi + f(\eta)\,,\\ u &= 1 + e^{-\xi} f(\eta)\,,\\ u &= 1 + e^{-x} f(y-2x)\,.\end{aligned}$$

Now we apply the initial condition

$$u(0,y) = 1 + f(y) = y^2$$

So $f(y) = y^2 - 1$ and the solution is given by,

$$u = 1 + e^{-x}((y-2x)^2 - 1)\,.$$

(b) Solve the equation $yu_x + xu_y = 0$ with $u(0,y) = e^{-y^2}$.
We can write the equation as $\nabla u \cdot (y,x) = 0$, so the characteristics must satisfy $\frac{dy}{dx} = \frac{x}{y}$. The solution is $c = x^2 - y^2$, so the general solution is given by $u = f(x^2 - y^2)$. We apply the initial condition to get $u(0,y) = f(-y^2) = e^{-y^2}$, so $f(y) = e^y$ and the solution is given by

$$u = e^{x^2 - y^2}\,.$$

2. Find the regions in the xy plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2 u_{yy}$$

is elliptic, hyperbolic or parabolic. Sketch them.

The discriminant is given by

$$\begin{aligned}B^2 - AC &= x^2y^2 + y^2(1+x)\,,\\ &= x^2y^2 + y^2 + xy^2\,,\\ &= y^2(x^2 + x + 1)\,.\end{aligned}$$

We note the roots of the quadratic in x are given by

$$x_{1,2} = \frac{-1+3i}{2}\,.$$

Since there are no real roots, we have $x^2+x+1>0$ for all real x. Thus the discriminant is zero when $y=0$ and positive elsewhere. So the equation is parabolic on the y-axis and hyperbolic elsewhere. Since the sketch is quite simple I'll skip it here.

3. Consider the equation,

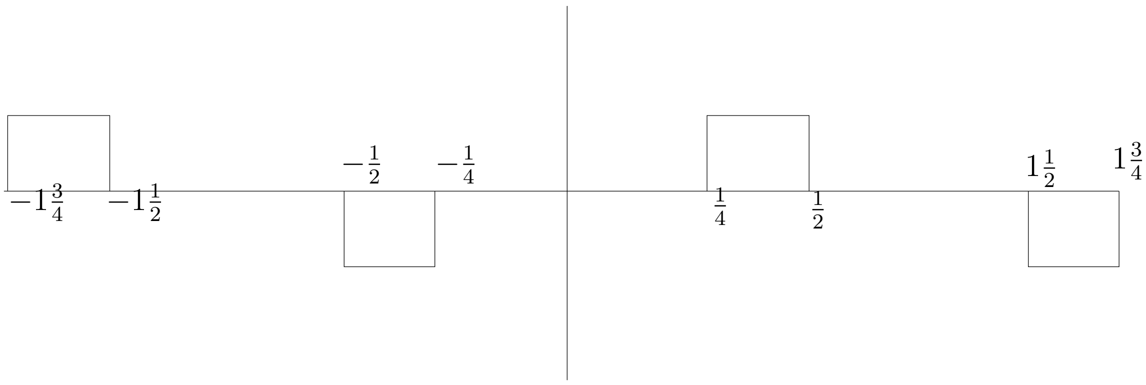
$$\begin{aligned} u_{tt} &= u_{xx}\,, \quad 0<x<1\,, \\ u(0,t) &= u(1,t) = 0\,, \\ u(x,0) &= \begin{cases} 0 & 0<x<\frac{1}{4}\,, \\ 1 & \frac{1}{4}<x<\frac{1}{2}\,, \\ 0 & \frac{1}{2}<x<1\,, \end{cases} \\ u_t(x,0) &= 0\,. \end{aligned}$$

Sketch the profile of the solution at $t=\frac{1}{4}$ and $t=\frac{1}{2}$.

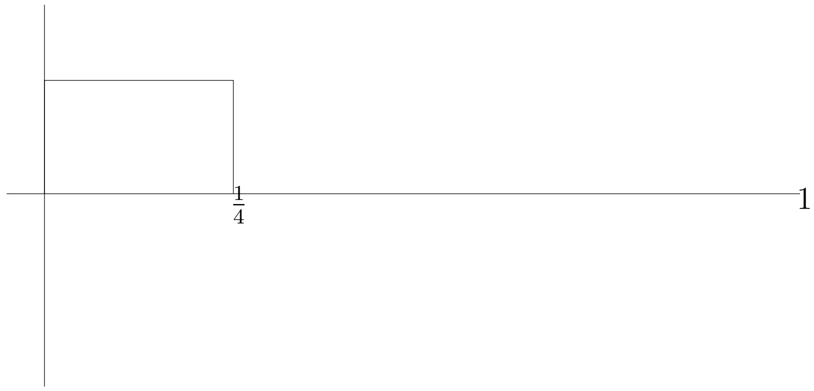
The solution is given by,

$$u = \frac{1}{2}(f(x+t)+f(x-t))\,,$$

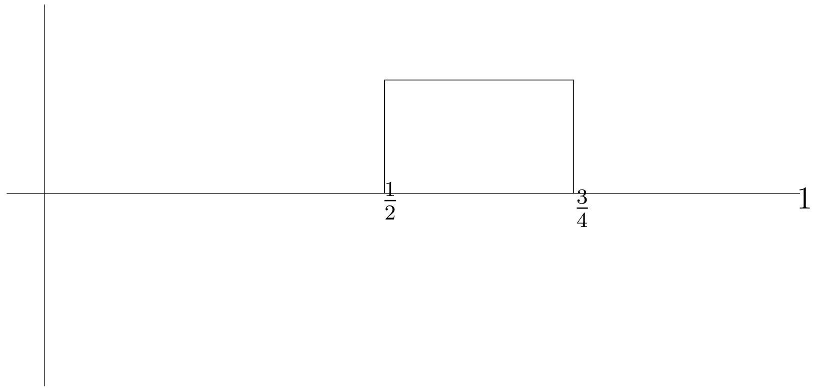
where f is the function obtained by first constructing an odd extension of $u(x,0)$ to $x\in[-1,1]$ then extending this to a 2-periodic function. Here is a graph of $f(x)$



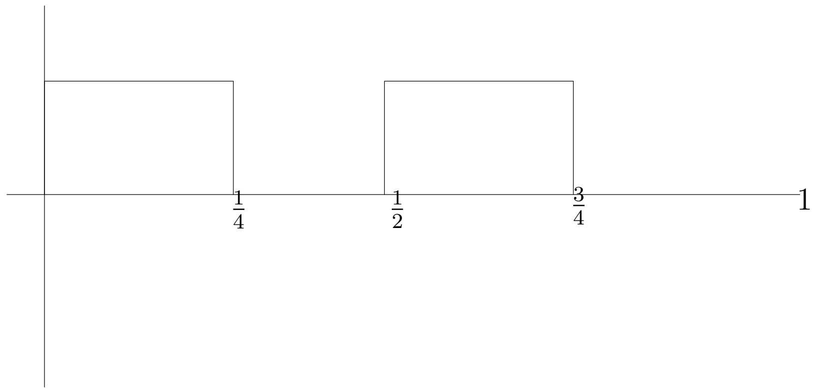
Now we will look at what happens where $t=\frac{1}{4}$.



(a) $f(x+\frac{1}{4})$



(b) $f(x-\frac{1}{4})$



(c) $u(x,\frac{1}{4})$

We repeat the procedure at $t = \frac{1}{2}$. We note that since $f(x) = 0$ for $\frac{1}{2} < x < 1\frac{1}{2}$, $f(x + \frac{1}{2}) = 0$ for $0 < x < 1$. So the solution here is just given by $\frac{1}{2}f(x - \frac{1}{2})$.

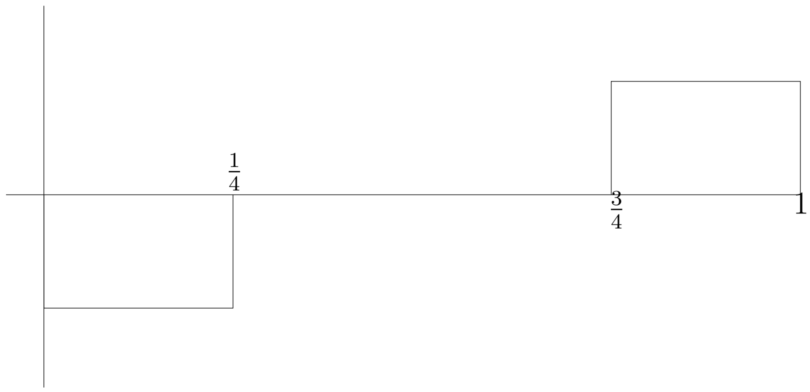


Figure 1: $u(x, \frac{1}{2})$

4. The motion of a vibrating string under the influence of air resistance is given by the following damped wave equation,

$$u_{tt} + 2cu_t - c^2 u_{xx} = 0\,, \quad 0 < x < \pi, \quad t > 0\,,$$

where u is the vertical displacement of the string at x at time t and $c > 0$ is a constant. The ends of the string are held fixed ($u(0,t) = u(\pi,t) = 0$). The initial displacement of the string is given by, $u(x,0) = 2\sin(2x)$ and the string is released with 0 initial velocity. Find the solution $u(x,t)$.

We can use separation of variables. We guess the solution is of the form $u(x,t) = X(x)T(t)$ and we get,

$$\frac{T'' + 2cT'}{c^2 T} = \frac{X''}{X} = -\lambda^2\,.$$

Here we have chosen $-\lambda^2$ as the constant for the eigenvalue problem to have nontrivial solutions. From the X equation, we will get the following eigenpairs:

$$X_n(x) = \sin(nx)\,,$$
$$\lambda_n = n\,.$$

So the equation for T is then,

$$T'' + 2cT' + n^2c^2T = 0\,.$$

The general soltution to this equation is just,

$$T_n = Ae^{(-c+c\sqrt{1-n^2})t} + Be^{(-c-c\sqrt{1-n^2})t}\,.$$

For $n = 1$ we have a double root, so,

$$T_1 = A_1e^{-ct} + B_1te^{-ct}\,,$$

and for $n > 1$ we get complex roots, so we have,

$$T_n = e^{-ct}(A_n\cos(c\sqrt{n^2-1}t) + B_n\sin(c\sqrt{n^2-1}t)\,.$$

The solution to PDE is then,

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x) \, .$$

Now we can use the initial conditions to solve for A_n and B_n. Since $u(x,0) = 2\sin(x)$, $A_n = 0$ and $B_n = 0$ for all $n \neq 2$. For the case $n = 2$ we first use $u(x,0) = 2\sin(2x)$ which gives us that $A_n = 2$. We now can use $u_t(x,0) = 0$ to find B_n. This gives us,

$$-cA_2 + c\sqrt{3}B_2 = 0 \, .$$

So $B_2 = \frac{2}{\sqrt{3}}$. We can write the solution as,

$$u = e^{-ct}(2\cos(c\sqrt{3}t) + \frac{2}{\sqrt{3}}\sin(c\sqrt{3}t))\sin(2x) \, .$$

5. Show that there are no solutions of

$$\Delta u = f \, , \text{ in } D \, , \frac{\partial u}{\partial n} = g \, , \text{on bdy } D$$

in three dimensions, unless

$$\iiint_D f \, dx \, dy \, dz = \iint_{\text{bdy(D)}} g \, dS \, .$$

For this question, we use Green's identity.

$$\begin{aligned} \iiint_D f \, dA &= \iiint_D \Delta u \, dA \, , \\ &= \iiint_D \nabla \cdot \nabla u \, dA \, , \\ &= \iint_{\partial D} \nabla u \cdot \vec{n} \, dS \, , \\ &= \iint_{\partial D} \frac{\partial u}{\partial n} \, dS \, , \\ &= \iint_{\partial D} g \, dS \, . \end{aligned}$$