Topics in Graph Theory – Problem set 2: Solution outlines.

(*A graph is k-critical if its chromatic number equals k, but removing any edge drops the chromatic number.*)

1. In class, we say Micielski's construction of how to find a sequence of graphs G_k which are triangle-free and have increasing chromatic number: $\chi(G_k) = k$. Show that each G_k is k-critical.

   Use induction. Assume G_k is k-critical, and form G_{k+1} with vertex set $V \cup U \cup \{w\}$ as described. Consider any edge e in G_{k+1}. Case 1: both endpoints of e are in V. Strategy: use $k-1$ colours on V, k-th colour on U, colour 1 on w. Case 2: e has one endpoint v in V, other u in U. Strategy: let v' be the vertex in V that u is a copy of. The subgraph induced by V with v' replaced by u is isomorphic to G_k. Removing e drops the chromatic number of this graph to $k-1$. Use colour k on vertex v'.Colour all other vertices in U with the same colour as the vertex they are a copy of. Colour w with colour k. Case 3: e has endpoint w, and an endpoint $u \in U$. Strategy: let v' be the vertex in V that u is a copy of. Colour $V - v'$ with $k-1$ colours, v' with colour k. Give U the colours of the vertices they copied from. Colour k only occurs on u, so it can be used on w.

2. Show that a k-critical graph cannot have a cut vertex.

   If the graph has a cut vertex v we can take the components of $G - v$ combined with v and colour them with less colours, and then combine these colouring as shown in class.

3. An *orientation* of a graph G is an assignment of a direction to each edge of G, where one of its endpoint is designated the *head*, and the other the *tail*. And orientation is *acyclic* if there are no directed cycles, so cycles where the orientations of the edges are consistently head-tail. Suppose that G has an acyclic orientation such that for every vertex v of G, at most k edges have v as its tail. Show that $\chi(G) \leq k+1$.

   Sorry for the earlier mistake! Order the vertices in a manner consistent with the orientation. So if an edge has head u and tail v, then u comes before v in the ordering. This gives an ordering where each vertex has at most k neighbours which come before it in the ordering. Such an ordering can always be found: note that an acyclic orientation must *always contain vertices who are not the tail of any edge. Call such a vertex a source. Find a source, put it first in the ordering, remove it, find a source in the remaining graph, put it second, etc.*

4. (MATH 4330/CSCI 4115) A *circular arc graph* is a graph were each vertex is associated with an arc on a circle, and vertices are adjacent precisely when the arcs overlap. (You can think of the arcs as time intervals in a periodic schedule.) (a) Give an upper bound on the chromatic number of a circular arc graph in terms of its clique number (so a statement of the form: $\chi(G) \leq a\omega(G)$ for each circular arc graph G). (b) Give an example of a circular arc graph where $\chi(G)/\omega(G)$ is as large as possible.

   $a = 2$. *Namely, any neighbour of an arc starting at a point s and ending at a point t around a circle must be an arc containing either s or t. All arcs containing s form a clique, as do all arcs containing t.*

5. Show that any 3-critical graph must be an odd cycle.

   A graph with chromatic number 3 is not bipartite, so it must contain an odd cycle. If the graph is 3-critical, then the odd cycle must be the graph itself.

6. (a) Construct a 4-critical graph with 6 vertices. (b) Describe an infinite family of 4-critical graphs.

   Wheels: an odd cycle with one extra vertex adjacent to all vertices of the cycle.

7. (a) What is the maximum number of edges in a graph with chromatic number k and size n? (b) (MATH 5330 only) Use (a) to deduce a lower bound on the chromatic number of a graph. Explain your answer.

   (a) Assuming k divides n, the answer is $\binom{k}{2}\frac{n^2}{k^2}$. Namely, take all colour classes of equal size, and add all possible edges between colour classes. (b) $m \leq \frac{n^2}{2} - \frac{n^2}{2k}$, so $k \geq \frac{n^2}{n^2-2m}$.

8. A *unit disk graph* is defined as follows: Vertices are disk with diameter one in the plane, and vertices are adjacent if the disks intersect in more than a point. An alternative definition is to say that vertices are points in the plane, and two vertices are adjacent if they are at distance less than 1 of each other.

(a) Consider a unit disk graph where the centers of the disks fall within strip of width a in the plane. Find the maximum value of a for which this graph is perfect.

$a = \frac{1}{2}\sqrt{3}$.

(b) Give a method of colouring a unit disk graph in a strip of width a (as above) perfectly.

Order vertices according to the location of their projection on the edge of the strip (imagine moving a ruler perpendicular to the strip from left to right, and order vertices in order that they are touched by the ruler). Then for each vertex, the neighbours that come before it in the strip form a clique. Thus this is a perfect elimination ordering.

(c) (MATH 5330) Use the above to give a bound on the ratio $\chi(G)/\omega(G)$ for all unit disk graphs G. Prove your answer.

3. Namely, we can divide the plane in parallel strips of width $\frac{1}{2}$. Then the subgraph induced by points in each strip is perfect, so can be coloured with ω colours. Moreover, these colours can be re-used every third strip. So in total we use at most 3ω colours. Note that the bound given in class gave the same ratio with a different method.