Topics in Graph Theory – Problem set 1

Due Thursday, Jan. 16, beginning of class

Write a careful argument to prove your answer. Be neat and precise. Try to make your argument as short as possible, but no shorter. Unmarked problems should be done by all students. Problems marked with a course number (e.g. MATH 4330) should be done by students enrolled in that course only.

1. Given a set of lines in the plane with no three meeting at a point, a graph G is formed whose vertices are the intersections of lines, with two vertices adjacent if they appear consecutively on one of those lines.

   Recall that a proper vertex colouring of a graph is an assignment of colours to the vertices so that adjacent vertices (connected by an edge) have different colours.

   Show that, for a graph G formed from lines in the plane as described above, there always exists a proper vertex colouring that uses only three distinct colours.

2. (a) Show that, in a k-colouring of a graph with chromatic number k, every colour class contains a vertex which is adjacent to vertices of every other colour. (b) Deduce that each graph with chromatic number k has at least k vertices of degree at least $k-1$.

3. Prove or disprove: every k-chromatic graph G has a proper k-colouring in which some colour class has $\alpha(G)$ vertices.

4. Prove or disprove: for every graph G, $\chi(G) \leq n(G) - \alpha(G) + 1$. Recall that $\chi(G)$ is the chromatic number, and $\alpha(G)$ is the independence number.

5. Read the wiki entry on the Erdös-Faber-Lovasz conjecture (see link on Course page). (a) For what amount is the award promised by Erdös for solving the conjecture? (b) Which mathematician got closest to the the solution of the conjecture? (Explain briefly)

6. A k-core of a graph is a maximal induced subgraph where each vertex has degree at least k. Show that the k-core is unique. Do not use the algorithm given in class, but use proof by contradiction: suppose there

are two distinct k-cores in a graph. Note that your proof justifies the fact that we talk about *the* k-core, and also that the algorithm given in class always leads to the unique k-core.

7. (MATH 4330/CSCI 4115) Give formulas for the independence number, clique number and chromatic number of the circulant graph $C(n, k)$, and prove your answer. *We did this in class, but here you are expected to give full proofs of your formulas.*

8. (MATH 5330) The notation $\delta(G)$ denotes the *minimum degree* of the graph G, and $H \leqslant G$ means that H is an induced subgraph of G. Show, that, for every graph G, $\chi(G) \leq 1 + \max_{H \leqslant G} \delta(H)$. (So the maximum of the minimum degree is taken over all induced subgraphs H of G.)