Topics in Graph Theory – Problem set 2

Due Thursday, Jan. 23, beginning of class

Write a careful argument to prove your answer. Be neat and precise. Try to make your argument as short as possible, but no shorter. Unmarked problems should be done by all students. Problems marked with a course number (e.g. MATH 4330) should be done by students enrolled in that course only.

(A graph is k-critical if its chromatic number equals k, but removing any edge drops the chromatic number.)

1. In class, we say Micielski's construction of how to find a sequence of graphs G_k which are triangle-free and have increasing chromatic number: $\chi(G_k) = k$. Show that each G_k is k-critical.

2. Show that a k-critical graph cannot have a cut vertex.

3. An *orientation* of a graph G is an assignment of a direction to each edge of G, where one of its endpoint is designated the l head, and the other the *tail*. And orientation is *acyclic* if there are no directed cycles, so cycles where the orientations of the edges are consistently head-tail. Suppose that G has an acyclic orientation such that for every vertex v of G, at most k edges have v as its tail. Show that $\chi(G) \leq k + 1$.

4. (MATH 4330/CSCI 4115) A *circular arc graph* is a graph were each vertex is associated with an arc on a circle, and vertices are adjacent precisely when the arcs overlap. (You can think of the arcs as time intervals in a periodic schedule.) (a) Give an upper bound on the chromatic number of a circular arc graph in terms of its clique number (so a statement of the form: $\chi(G) \leq a\omega(G)$ for each circular arc graph G). (b) Give an example of a circular arc graph where $\chi(G)/\omega(G)$ is as large as possible.

5. Show that any 3-critical graph must be an odd cycle.

6. (a) Construct a 4-critical graph with 6 vertices. (b) Describe an infinite family of 4-critical graphs.

7. (a) What is the maximum number of edges in a graph with chromatic number χ? (b) (MATH 5330 only) Use (a) to deduce a lower bound on the chromatic number of a graph. Explain your answer.

8. A *unit disk graph* is defined as follows: Vertices are disk with diameter one in the plane, and vertices are adjacent if the disks intersect in more than a point. An alternative definition is to say that vertices are points in the plane, and two vertices are adjacent if they are at distance less than 1 of each other.

   (a) Consider a unit disk graph where the centers of the disks fall within strip of width a in the plane. Find the maximum value of a for which this graph is perfect.

   (b) Give a method of colouring of colouring a unit disk graph in a strip of width a (as above) perfectly.

   (c) (MATH 5330) Use the above to give a bound on the ratio $\chi(G)/\omega(G)$ for all unit disk graphs G. Prove your answer.