Topics in Graph Theory – Problem set 7

Due Tuesday, March 11, beginning of class

1. Consider Polya's urn problem as presented in class on March 6, but now assume that at time 0 there are two white balls and one black ball in the urn. Let X_t be the proportion of white balls in the urn at time t. Compute the expected value $E(X_t)$. Use conditional expectation.

2. Consider the following variation of the preferential attachment model. The process builds a sequence of graphs G_t as follows. G_1 consists of a vertex and a loop at that vertex (so the degree of that vertex is one). At each time step $t > 1$, G_{t+1} is formed by adding one vertex, v_{t+1}, to G_t, and one edge from v_{t+1} to a vertex of G_t chosen according to a link probability proportional to $\deg(u,t) + a$ ($\deg(u,t)$ is the degree of . Here a is parameter of the model. This question asks you to analyze the degree distribution of this model in a way similar as done for the original PA model in class on March 6.

   (a) Precisely, for any $u \in V(G_t)$, the probability that v_{t+1} links to u equals $P(v_{t+1} \sim u) = c(\deg(u,t) + a)$. The constant c is determined by the fact that all probabilities have to add to 1, so

   $$\sum_{v \in V(G_t)} P(v_{t+1}, u) = 1.$$

   Determine c.

   (b) Use conditional expectation to find a recurrence relation for $\deg(v_i, t)$, where v_i is the vertex born at time i, and the recurrence should be in terms of t.

   (c) Use a DE-based method to find an approximate expression for $E(\deg(v_i, t)$. To do this, assume that the growing process ends at time $t = n$, and define the function f so that, for all $1 \leq i \leq t \leq n$,

   $$f\left(\frac{i}{n}\right) = \frac{\mathbb{E}(\deg(v_i, t))}{n}.$$

   (d) Use the expression found in (b) to find an approximate expression for $N_{\geq k}$, the number of vertices of degree at least k, where the assumption is that the degree of a vertex is approximately equal to its expected value.

   (e) Does the resulting graph have a power law degree distribution? If so, what is the exponent? Explain your answer.

   (f) What effect does the parameter a have on the degree distribution? What happens if a is negative?