**Topics in Graph Theory – Problem set 9**

*Due Thursday, March 27, beginning of class*

1. Consider the random geometric graph $RGG(n, r, d)$, where the dimension $d$ equals 1. Let $X$ be the number of induced paths $P_2$ (a path with three vertices and two edges). What is the probability that a given set of three vertices induces a $P_2$? Find $E(X)$ using indicator variables.

2. Show that geometric graphs in one dimension are perfect. Is the same true for higher dimensions?

3. Consider the copy (GNC) model presented by Melanie on March 13. In this model, a directed graph $G_t$ is created as follows: $G_1$ consists of one vertex $v_1$. At each subsequent time $t+1$, one vertex $v_{t+1}$ is created. A vertex $u$ is chosen uniformly at random from $G_t$, and edges from $v_{t+1}$ to $u$ and to all out-neighbours of $u$ in $G_t$. Melanie showed that possible graphs are the directed path $(v_1 \leftarrow v_2 \leftarrow \ldots \leftarrow v_t)$ and the directed complete graph. For each of these graphs, give the probability that it occurs. *Solve this first for small values of $t$, to get an idea.*

4. Consider the Kronecker model presented by Kyle on March 18. (a) Give the Kronecker product of the graph $C_4$ with the graph $P_2$. (Draw the graph and give its adjacency matrix). Assume both graphs have a loop at each vertex, so the adjacency matrix has ones on the diagonal. (b) Suppose $G$ is a connected graph, but $H$ is not connected. Is the Kronecker product always connected? Motivate your answer.

5. [MATH 4330/CSCI 4115] A random threshold graph $RTG(n, t)$ is defined as follows. It starts with $n$ vertices $v_1, \ldots, v_n$. Each vertex $v_i$ is assigned a value $x_i$ chosen uniformly at random from $[0, 1]$. Two vertices $v_i$ and $v_j$ are connected if and only if $x_i + x_j \geq t$. In class on March 20, we saw that such graphs can be represented at $w$-random graphs. Define the function $w$ that produces random threshold graphs, and justify your answer.

6. MATH 5330] The following model was presented in class by Cuong. The geographic threshold model $GTG(n, t, d)$ is defined as follows. Vertices $v_1, \ldots, v_n$ are chosen uniformly at random from $[0, 1]^2$, and each vertex $v_i$ is assigned a weight $w_i$ chosen uniformly at random from $[0, 1]$. Two vertices are adjacent if and only if
$$\frac{w_i + w_j}{r^2} \geq t, \text{ where } r = \text{dist}(v_i, v_j).$$
In class on March 20, we saw that many random graphs can be represented at $w$-random graphs. Define the function $w$ that produces geographic threshold graphs, and justify your answer. *You may assume that for any dimension $d$, there exists a bijection $\phi : [0, 1] \to [0, 1]^d$ so that, if $x$ is chosen u.a.r from $[0, 1]$, then $\phi(x)$ is a point chosen u.a.r. from $[0, 1]^2$. In technical terms, $\phi$ is measure preserving. Hint: you will need this result for $d = 3$.*