Topics in Graph Theory – 2

January 14 and 16, 2014

For interval graphs, we obtained a perfect colouring using a very special type of ordering, with the property that all uncoloured neighbours form a clique. This idea carries over to other classes of graphs.

For a graph $G = (V, E)$, a *perfect elimination ordering* of G is an ordering $v_1, v_2, \ldots, v_n$ so that, for each vertex v_i, $N(v_i) \cap \{v_1, \ldots, v_{i-1}\}$ is a clique.

Theorem 1. *If a graph G has a perfect elimination ordering then G is perfect.*

Proof. If $G = (V, E)$ has a perfect elimination ordering, $v_1, v_2, \ldots, v_n$, then this ordering has the property that, for each vertex v_i, $N(v_i) \cap \{v_1, \ldots, v_{i-1}\}$ has size at most $\omega(G) - 1$. Thus, the greedy colouring uses at $\omega(G)$ colours, and $\chi(G) = \omega(G)$. To complete the proof, note that if G has a perfect elimination ordering, then so does each induced subgraph of G. □

A *subgraph* of a graph $G = (V, E)$ is a graph $H = (V_H, E_H)$ so that $V_H \subseteq V_G$ and $E_H \subseteq E_G$. A subgraph H is an *induced* subgraph if every edge in E_G with endpoints in V_H is included in E_H. A subgraph H is a *spanning* subgraph if $V_H = V_G$. An *induced cycle* is an induced subgraph which is a cycle. This means that the cycle has no *chords*, i.e. no other edges than the cycle edges connecting the vertices of the cycle. A graph $G = (V, E)$ is *chordal* if it has no *induced* cycles of size larger than 3.

By definition, interval graphs are perfect, but the converse is not true.

Theorem 2. *A graph G is chordal if and only if it has a perfect elimination ordering.*

Before we give the proof of this theorem we need a few lemmas.

Theorem 3. *Every connected graph contains at least two vertices which are not cut vertices.*

These vertices can be found by taking a maximal path in a graph, and taking its endpoints.

Lemma 4. *Every graph chordal graph G has a vertex such that its neighbourhood is a clique.*

Proof. Assume without loss of generality that G is connected. (If not, take a connected component). Let v be a vertex which is not a cut vertex. We will argue that the neighbours of v form a clique. Assume the contrary. This means that there exist two neighbours u, w of v which are not adjacent. Since v is not a cut vertex, $G - v$ is connected, so there exists a path from u to w in $G - v$. Let P be a shortest such path. Then by extending P with v we find an induced cycle of size larger than 3. Thus, G is not chordal. □

Proof of Theorem 2. Assume first that G is chordal. Then so is each induced subgraph of G. By the lemma, G contains a vertex v_n whose neighbourhood is a clique. Put this vertex last in the ordering. Then find a vertex v_{n-1} in $G - v_n$ whose neighbourhood in $G - v_n$ is a clique. Continuing this process, this results in a perfect elimination ordering $v_1, \ldots, v_{n-1}, v_n$.

Next, assume G is not chordal. Then G must have an induced cycle C of size $k > 3$. Now consider any ordering of the vertices of G. Let v be the vertex of C that comes last in the ordering. This means that the two neighbours u and w on C of v come before v in the ordering. However, since C is induced, u and w are not connected. Thus the ordering is not a perfect elimination ordering. □

We can extend the idea of a perfect elimination ordering to obtain bounds on the ration χ/ω. Suppose a graph $G = (V, E)$ has an ordering $v_1, v_2, \ldots, v_n$ so that, for each vertex v_i, $N(v_i) \cap \{v_1, \ldots, v_{i-1}\}$ can be partitioned into at most k cliques. Then v has at most $k\omega - 1)$ neighbours in $N(v_i) \cap \{v_1, \ldots, v_{i-1}\}$, and thus the greedy colouring using this ordering will use at most $k\omega$ colours. This implies that $\chi(G) \leq k\omega(G)$.

A graph $G = (V, E)$ is a *geometric graph* if the vertices can be embedded in ^{2} so that two vertices are adjacent if and only if they have distance at most t from each other, where t is a given threshold value. Geometric graphs are also called *unit disk graphs*.

Theorem 5. *If G is a geometric graph, then $\chi(G) \leq 3\omega(G)$.*

Proof. Note first that all neighbours of a vertex of G lie inside a circle with radius t. Order all vertices from left to right, i.e. according to increasing x-coordinate. (If two vertices have the same x-coordinate, then give preference to the smallest y coordinate.) Then for each vertex v with coordinates (x_v, y_v), the vertices coming before v in the ordering lie in the half circle with radius t around v, which lies to the left of the vertical line through (x_v, y_v).

This half-circle can be partioned into three equal regions, each of which has diameter t. Thus all vertices in one of these regions form a clique. Thus the neighbours of v that come before it in the ordering can be partioned into 3 cliques. □

A graph G is k-critical if each proper induced subgraph H of G (i.e. every induced subgraph except G itself) has $\chi(H) < \chi(G)$. Every k-chromatic graphs has a k-critical subgraph. In every k-chromatic graph, every vertex has degree at least $k-1$.

Theorem 6. *In a k-critical graph ($k \geq 2$), no vertex cut is a clique.*

Proof. Suppose, by contradiction, that G is k-critical, has a vertex cut C that is a clique. Let $H_1, \ldots, H_\ell$ be the components of $G - C$, and, for each i, let G_i be the graph formed from H_i by adding C and all edges between vertices in C and H_i. Each G_i is a proper subgraph, and can therefore be coloured with $k-1$ colours. In such a colouring, each vertex in C must receive a different colour. *Wlog*, we can adjust the colourings of the G_i so that the colour of each vertex in C is the same in the colouring of each G_i. But since the G_i only overlap in C, this leads to a colouring of G. □

A corollary of this theorem is that no k-critical graph can have a cut vertex.

The concept of critical graphs is used in the proof of Brook's theorem. First, note that the only 1-critical graph is K_1, and the only 2-critical graph is K_2. The only 3-critical graphs are odd cycles (problem set 3).

Theorem 7 (Brook's theorem)**.** *. If G is a connected, simple graph which is neither an odd cycle or a clique, then $\chi(G) \leq \Delta(G)$.*

Proof. Let G be a k-chromatic graph that satisfies the hypothesis of the theorem. By the statements above, this implies that $k \geq 4$. We may assume *wlog* that G is k-critical.

Case 1 : G has a vertex cut of size 2, say $\{u, v\}$. Since G is k-critical, by Theorem 3, u and v are not adjacent. Moreover, if $G_1, \ldots, G_\ell$ are defined as in the proof of Theorem 3, then there must be two of the G_i, say G_1 and G_2, so that every $k-1$-colouring of G_1 assigns u and v different colours, while any $k-1$-colouring of G_2 assigns u and v the same colour. This mean that, in G_1, any colour on a neighbour of u is assumed by a neighbour of v. Therefore, u and v together have at least $k-1$ neighbours in G_1. On the other hand, u and v each must see all colours except one in G_2, so u and v together have at least $2(k-2)$ neighbours in G_2. Therefore, u and v together must have at least $3k - 5 \geq 2k - 1$ neighbours in G, which means that at least one of u, v must have degree k, so $\Delta(G) \geq k$, as required.

Case 2: G does not have a vertex cut of size 2 (G is 3-connected). Since G is not a clique, there must be three vertices u, v, w so that u, v and v, w are adjacent, but u, w are not. Set $u = v_1$ and $w = v_2$. Order the vertices in $G - \{u, w\}$ according to non-increasing graph distance from v, so if $i < j$ then $dist(v_i, v) \geq dist(v_j, v)$. Note that this implies that $v_n = v$ $(n = n(G))$, and for each $i \geq 4$, v_i has at least one neighbour v_j with $j > i$. Now colour the vertices with the greedy colouring algorithm using this ordering. Then v_1 and v_2 will both receive colour 1. All vertices $v_3, \ldots, v_{n-1}$ have at least one uncoloured neighbour, and thus at most $\Delta - 1$ coloured neighbours, so these can be coloured with Δ colours. Now $v_n = v$ has two neighbours v_1 and v_2, with the same colour, so v_n is adjacent to at most $\Delta - 1$ colours, and the colouring can be completed with Δ colours. Thus, $k \leq \Delta$ □