Topics in Graph Theory – 3
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Instead of colouring vertices, we can also colour the edges of a graph. A *proper edge colouring* of a graph $G = (V, E)$ is an assignment of one colour to each edge in E so that incident edges receive different colours. The *chromatic index* $\chi'(G)$ of a graph is the minimum number of colours required for a proper edge colouring of G.

Vertex and edge colourings are related through the concept of line graphs. Given a graph $G = (V, E)$, the *line graph* $L(G)$ is the graph with vertex set E, and two vertices of $L(G)$ are adjacent if the edges they represent are incident in G. The collection of all edges incident with a given vertex form a clique in $L(G)$. On the other hand, the edge of a triangle in G also form a clique in $L(G)$. These are the only cliques in $L(G)$. Thus, $\Delta(G) = \omega(L(G))$ unless G is a collection of disjoint triangles. Clearly, $\chi'(G) = \chi(L(G))$. An induced subgraph of $L(G)$ is the line graph of a subgraph (not necessarily induced) of G. Thus if a graph G which is not a collection of disjoint triangles has the property that for every subgraph H of G, $\chi'(G) = \Delta(G)$, then $L(G)$ is perfect.

A colour class of an edge colouring of G will be a set of edges so that no two of them are incident. Such a set is called *matching*. (A matching is the equivalent of an independent set in the case of vertex colourings, and sometimes is referred to as an *independent set of edges*.)

A *matching* in a graph G is a set of edges with no shared endpoints. The vertices incident to the edges of a matching M are *matched* by M; the others are *unsaturared* (we also say M-matched and M-unmatched). A *perfect matching* in a graph is a matching that matches every vertex. A *maximal matching* in a graph is a mathcing that cannot be enlarged by adding another edge. A *maximum matching* is a matching of maximum size among all matchings in the graph. Note that every maximum matching is maximal, but the converse is not true.

given a matching M, an M-alternating path is a path that alternates between edges in M and edges not in M. An M-alternating path whose endpoints are unmatched by M is an M-augmenting path.

Any M-augmenting path P can be used to find a larger matching. Namely, remove all M-edges on P from M, and instead add all non-M-edges. Because the endpoint of P are M-unmatched, the new matching M' is still a matching, every vertex that was M-matched is also M'-matched, and the endpoints of P are M'-matched. Since P starts and ends with a non-M edge, there are more non-M edges than M-edges in P, so M' is one edge larger than M.

Thus, if there is an M-augmenting path, M is not a maximum matching. By an result from Berge (1957), the converse is also true.

Theorem 1. *A matching M in a graph G is a maximum matching if and only if G has no M-augmenting path.*

Proof. We saw, by the argument above, that if a matching M is a maximum matching then G has no M-augmenting path.

Suppose M is not a maximum matching. Then there is a matching, M', which is larger than M'. Consider the set S of edges that belong to M or M', but not both (this is called the symmetric difference of M and M'. Since both M and M' are matchings, in the graph H formed by the edges in S and their endpoints, every vertex has degree at most two. Therefore, H is a collection of disjoint paths and cycles. Moreover, every cycle and path must be M-alternating, and therefore every cycle must be even. Since M' is larger than M, H must contain more M'-edges than M-edges. Therefore, there must be a components of H that contains more M'-edges than M-edges. This component must be a path, say P. Since P has more M' edges than M-edges, P must be of odd length, and start and end with an M-edge. Moreover, since P is a maximum connected component of H, the endpoints of P must be M-unmatched. Thus, P is an M-alternating path whose endpoints are M-unmatched, and thus P is an M-augmenting path. □

The theorem gives a key to an algorithm to find a maximum matching: start with any matching M. Try to find an M-augmenting path. If such a path exists, then construct a larger matching. Repeat the process until no more augmenting paths can be found, at which point the matching is maximum.

A lower bound on the maximum size of a matching is the minimum size of a vertex cover. A *vertex cover* of a graph G is a set S of vertices so that every edge of G has at least one endpoint in G. Since a matching is a subset of edges of a graph, and since for every edge in a matching we need a different vertex in the vertex cover, we have that the minimum size of a vertex cover is greater than or equal to the maximum size of a vertex cover. In fact, if a graph has a matching and a vertex cover of equal size, then we know that the matching is maximum and the vertex cover is minimum. We sometimes refer to the vertex cover of the same size as a *certificate of optimality* for the maximum matching, i.e. providing a matching together with a vertex cover of the same size provides proof that the matching is, in fact, maximum.

It is not always possible to find such a vertex cover/matching pair; in odd cycles (length $2k+1$), for example, the minimum size of a vertex cover is $k+1$, while the maximum size of a matching is k. However, in bipartite graphs we do always have equality. This result is known as the *König-Egerváry theorem*.

Theorem 2. *In every bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.*

Proof. Let G be a graph with bipartition (X, Y), and let M be a maximum matching. We only need to show that we can find a vertex cover of size $|M|$. If M matches all vertices in X, then $|M| = |X|$, so X itself is a vertex cover of size equal $|M|$.

Suppose M does not match all vertices in X. Let X_0 be the set of all unmatched vertices in X, and let X_1 and Y_1 be the set of all vertices in X and Y, respectively, that can be reached by alternating paths starting at a vertex in X_0. Since M is maximum, there are no augmenting paths, so every path must end with a matched vertex in X_1. Hence also, all vertices in Y_1 are matched to vertices in X_1, so $|X_1| = |Y_1|$. Also, all neighbours of vertices in $X_0 \cup X_1$ are in Y_1. Therefore, every edge in G either has an endpoint in Y_1, or in $X - (X_0 \cup X_1$. Let A be the set consisting of all vertices in $X - (X_0 \cup X_1)$ or in Y_1. By the above, A is a vertex cover. Moreover, every vertex in A is matched, so $|A| = |M|$. $\square$

A *perfect matching* is a matching that matches all vertices. Clearly, a necessary condition for a graph with bipartition (X, Y) to have a perfect matching is that $|X| = |Y|$. If a bipartite graph is k-regular, then a simple counting argument shows that the two sides of the bipartition must have equal size. The following theorem shows that such graphs also always have a perfect matching.

Theorem 3. *Every regular bipartite graph has a perfect matching.*

Proof. Let G be a bipartite, k-regular graph with bipartion (X, Y). Let $m = |X| = |Y|$. Suppose that G does not have a perfect matching, i.e. the size of a maximum matching is less than m. This means that G must have a vertex cover A of size less than m. Let $A_X = A \cap X$ and $A_Y = A \cap Y$. Since A is a vertex cover, there can be no edge from $X - A_X$ to $Y - A_Y$, so all neighbours of vertices in $X - A_X$ must be in A_Y. Also, since $|A| = |A_X| + |A_Y| < |X|$, we have that $|A_Y| < |X - A_X|$. However, there are $k|X - A_X|$ edges going out of $X - A_X$, and all of them are going to A_Y, so one of the vertices of A_Y must have degree greater than k, which is a contradiction. $\square$

As a corollary, we have that k-regular bipartite graphs can be edge-coloured with k colours. Clearly, this is best possible.

For every regular bipartite graph G, $\chi'(G) = \Delta(G)$.

In fact, the corollary holds for all bipartite graphs. This follows from two observations. The first observation is that all theorems presented here hold also for *multigraphs*, so graphs where multiple edges between the same pair of vertices are allowed. The second observation is that each bipartite graph G can be turned into a bipartite Δ-regular multigraph by adding edges. This graph can then be edge-coloured with Δ colours, which also gives an optimal edge colouring of the original graph.

For every bipartite graph G, $\chi'(G) = \Delta(G)$.

Since all subgraphs of a bipartite graph are also bipartite graphs, we have that all induced subgraphs of the linegraph of a bipartite graph is also the linegraph of a bipartite graph, and thus has chromatic number equal to the clique number.

All line graphs of bipartite graphs are perfect.