Topics in Graph Theory – 4
January 23 and 28, 2013

In many applications, it is necessary to assign more than one colour to a vertex or edge. We can extend the idea of colouring: given a graph G and a *demand* $s(v)$ for each vertex v, a multicolouring of (G, s) is a assignment of a set of $s(v)$ distinct colours to each vertex v, so that the colour sets on adjacent vertices are disjoint. We will refer to the minimum number of colours needed as $\chi(G, s)$.

The problem of multicolouring can be turned into a regular vertex colouring of a related graph. Given a graph G and demands $s(v)$, replace every vertex v of G by a clique of size $s(v)$. Replace every edge uv in G with edges from every vertex in the clique replacing u to the clique replacing v.

Clearly, a lower bound on the number of colours needed for a multicolouring of graph G with demand vector s is given by the *weighted clique number* $\chi(G, s)$, which is the maximum, over all cliques C of G, of the sum of the demands on all the vertices of C. So

$$\chi(G, s) = \max_{C \text{ clique}} \sum_{v \in C} s(v).$$

Since every colour can still only be used at most $\alpha(G)$ times, we have that

$$\chi(G, s) \geq \sum_{v \in V} s(v)/\alpha(G).$$

We will refer to the righthandside as the *weighted clique number* $\omega(G, s)$.

An edge multicolouring is defined in an analogous way, and the minimum number of colours needed to colour the edges of a graph G with edge weight vector s is referred to as $\chi'(G, s)$. A lower bound for the number of colours needed is then given by the *weighted degree*: the maximum, over all vertices v, of the demands on all edges incident with v. Here, a multicolouring of a graph G with edge demands $s(e)$ can be interpreted as an edge colouring of a graph where each edge e of G is replaced by $s(e)$ parallel edges. Since the König-Egervary theorem applies equally to graphs with parallel edges, we have that for any bipartite graph G, and any edge demand vector s,

$$\chi'(G, s) = \Delta(G, s).$$

Perfect graphs also behave perfectly with respect to weighted colourings. This beautiful result is due to Lovász, who proved this in 1972 at age 22. We will call a colouring of $G, s)$ with $\omega(G, s)$ colours a *perfect colouring*.

Theorem 1. *If G is perfect, then for any vector of non-negative integer demands s, (G, s) has a perfect colouring.*

Proof. Let $G = (V, E)$ be a perfect graph, and s a non-negative integer demand vector.

First, note that the case where $s(v) \leq 1$ is dealt with by the property that G is perfect: let H be the subgraph induced by all vertices with weight 1. Then $\omega(G, s) = \omega(H)$ and any colouring of H is a colouring of (G, s).

The remainder of the proof is by induction on $\sum_{v \in V} s(v)$, where the base case is given by the previous paragraph.

For the induction step, fix s, and let v be a vertex so that $s(v) \geq 2$. Consider the weight vector s' where $s'(v) = s(v) - 1$, and $s'(u) = s(u)$ for all other vertices. Since the sum of the weights given by s' is smaller than that given by s, by the induction hypothesis, (G, s') has a perfect colouring. Let $k = \omega(G, s)$. We distinguish two cases:

$\omega(G, s') = k - 1$, then we can assign to v a colour not occurring in the colouring of (G, s') to obtain a colouring of (G, s) that uses $(k - 1) + 1 = \omega(G, s)$ colours.

Assume then that $\omega(G, s) = \omega(G, s')$. Note that this implies that v does not belong to any maximum clique in (G, s'). In other words, the weight of any clique containing v in (G, s') is at most $k - 1$. By induction, (G, s') has a perfect colouring, which uses k colours. At least one of the colours classes includes v. Let this colour class be A. Now consider the demand vector s'_A, given as follows: $s'_A(u) = s'(u) - 1$ if $u \in A$, and $s'_A(u) = s'(u)$ otherwise. The remaining colours give a colouring of (G, s'_A) using $k - 1$ colours, so $\omega(G, s'_A) = k - 1$. Moreover, since $v \in A$, the weight of each clique containing v in s'_A is at most $k - 2$. Now consider the vector s_A, where $s_A(v) = s'_A(v) + 1$, and $s_A(u) = s'_A(u)$ for all other vertices. Since v was not contained in any maximum weight clique, we have that $\omega(G, s_A) = k - 1$. By induction, (G, s_A) can be coloured using $k - 1$ colours. Finally, note that $s_A(u) = s(u) - 1$ if $u \in A$, and $s_A(u) = s(u)$ otherwise. Therefore, if we add a new colour and assign it to the vertices of A, we obtain a colouring of (G, s) using k colours. □

Note that in a perfect colouring, every colour class must intersect every maximum clique. By an inductive argument, we can show the converse, resulting in the following theorem.

Theorem 2. *A graph G is perfect if every induced subgraph H has the property that H has an independent set which intersects every maximum clique.*

The *complement* $\bar{G}$ of a graph $G = (V, E)$ is the graph with vertex set V where two vertices u, v are adjacent in the complement precisely when they are not adjacent in G.

Lovász proved the theorem above as a lemma to prove the following theorem, conjectured by Berge.

Theorem 3. *A graph is perfect if and only if its complement is perfect.*

Proof. Note that the cliques in $\bar{G}$ are the independent sets of G, and vice versa. By Theorem 2, it suffices to show that every induced subgraph H of G has the property that H has a clique which intersects every maximum independent set.

Assume by contradiction that there is an induced subgraph H which does not have the property. Thus, for every clique of H there is an independent set which does not intersect it. Note that it is sufficient to consider only maximal cliques. Let $C_1, \ldots, C_K$ be the collection of all maximal cliques of H, and for each i, let A_i be a maximum independent set which does not intersect C_i. Now form a demand vector s as follows. For each $v \in V(H)$,

$$s(v) = |\{i : v \in A_i\}|$$

In other words, the demand of vertex v equals the number of sets A_i that it belongs to.

Now (H, s) has a colouring with K colours: give each A_i a different colour. Thus, $\chi(H, w) \leq K$. In fact, we have equality. Namely, since every set A_i is a maximum independent set, its size equals $\alpha(H) = \alpha$, so

$$\sum_v s(v) = \sum_v |\{i : v \in A_i\}| = \sum_i |A_i| = K\alpha.$$

Thus, $\chi(H, s) \geq (K\alpha)/\alpha = K$, so the colouring given is best possible.

Note that any maximum demand clique in (H, s) must be a maximal clique in G, and thus be one of the C_i. For any i,

$$\sum_{v \in C_i} s(v) = \sum_{v \in C_i} |\{i : v \in A_i\}| = \sum_j |C_i \cap A_j|.$$

Since an independent set can intersect a clique in at most one vertex, we have that $|C_i \cap A_j| \leq 1$ for all j. Moreover, we know by definition that $|C_i \cap A_i| = 0$. Thus,

$$\sum_{v \in C_i} s(v) \leq K - 1$$

for all i, and thus $\omega(G, w) \leq K - 1$. This contradicts Theorem 1.

$\square$