Topic in Graph Theory

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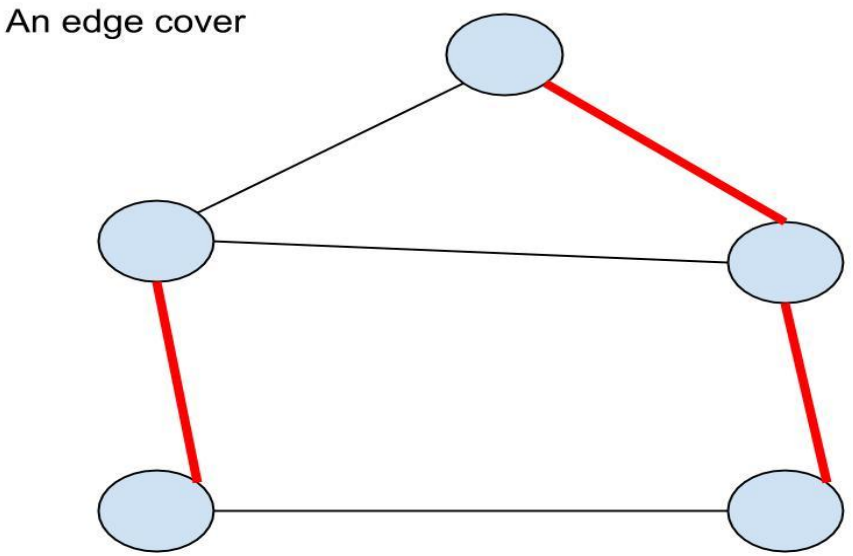
Preliminaries

Given graph $G = (V, E)$

$\alpha'(G)$: matching number or the size of the maximum matching.

An edge cover of G is a subset of E which covers all vertices in V. Denote $\beta'(G)$ as the size of the minimum edge cover.



3.1.22 Theorem Gallai [1959]

G: graph without isolated vertex (vertex with degree 0), then $\alpha'(G)+\beta'(G) = n(G)$.

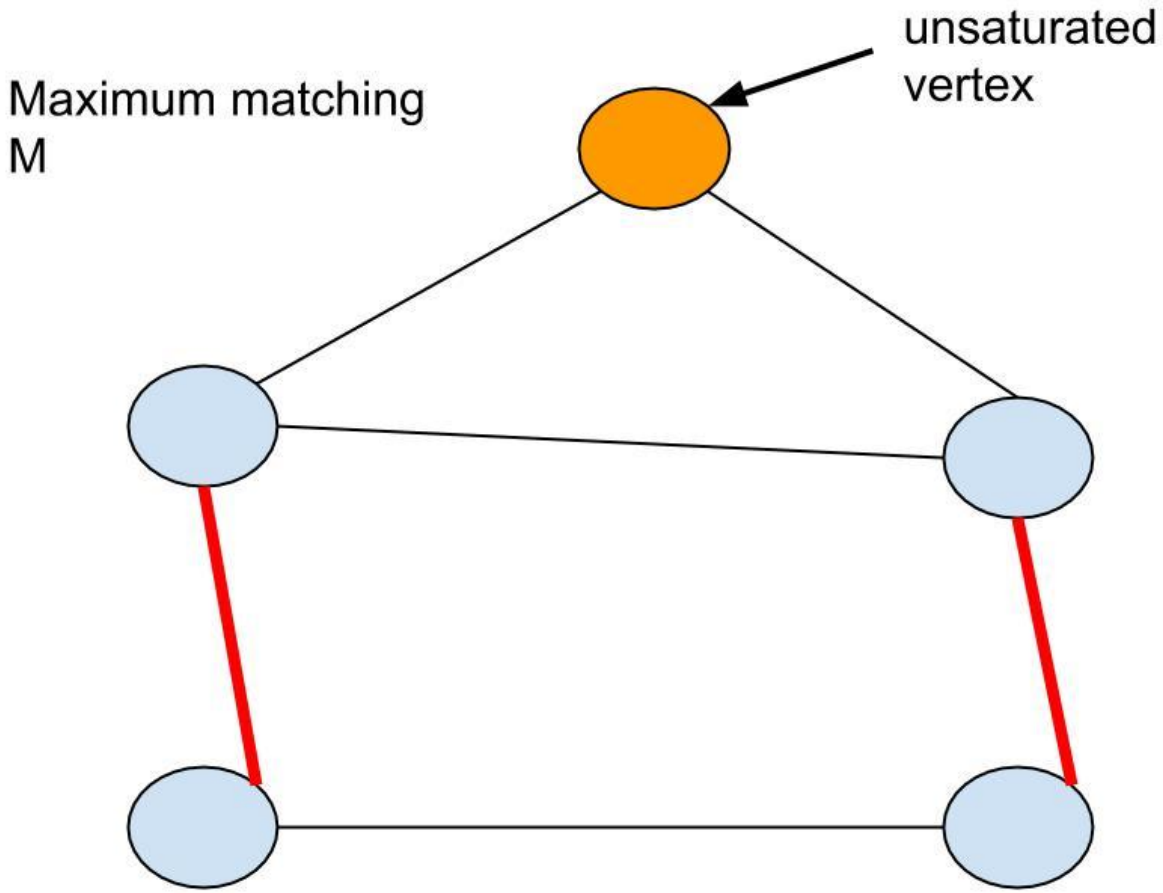
The requirement for no isolated vertex is trivial because how do we have an edge cover if exists an isolated vertex ?

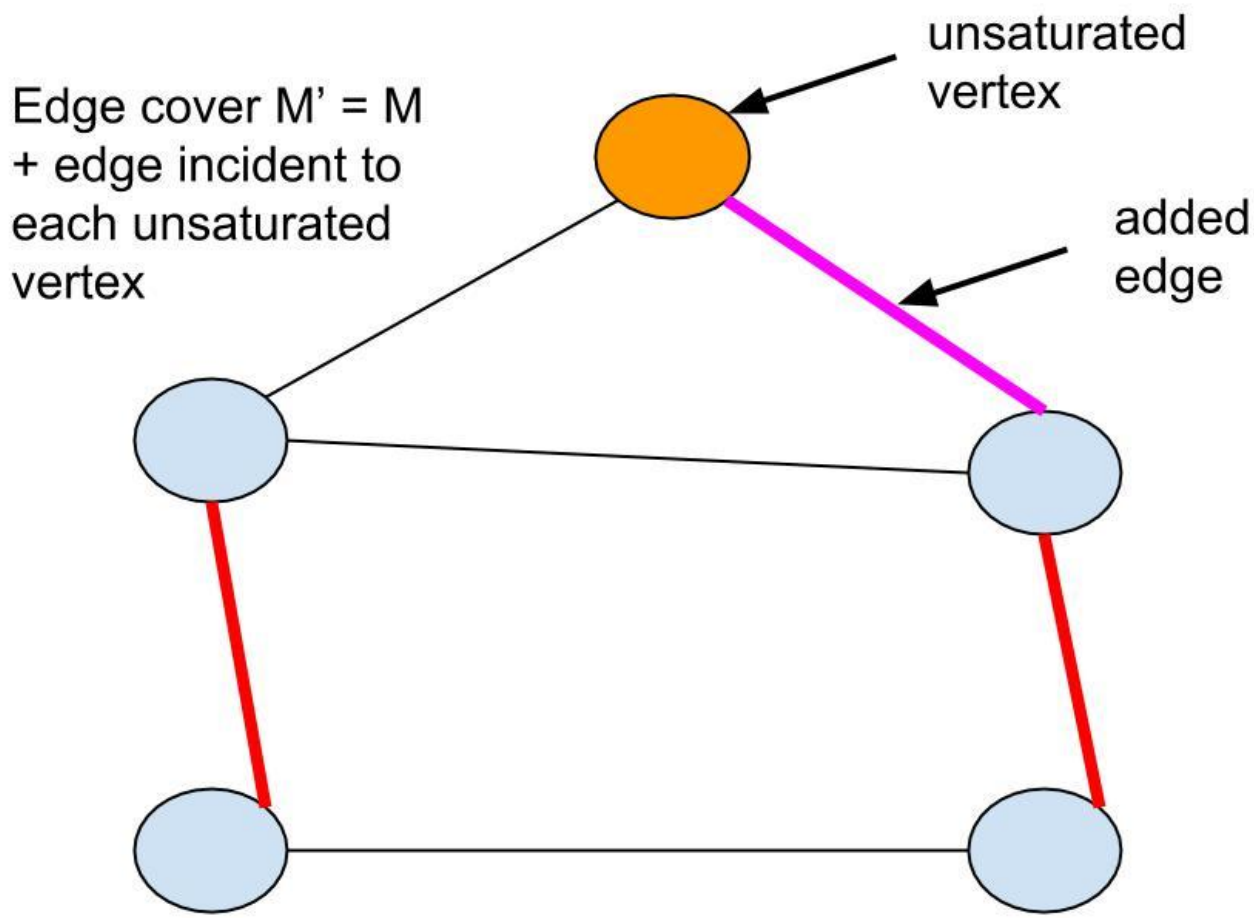
Proof

Strategy: 1) Prove $\alpha^{'}(G) + \beta^{'}(G) \leq n(G)$
Given a maximum matching, we construct an edge cover with size $n(G) - \alpha^{'}(G)$.
2) Prove $\alpha^{'}(G) + \beta^{'}(G) \geq n(G)$
Given a minimum edge cover, we construct a matching with size $n(G) - \beta^{'}(G)$.

1) Prove $\alpha'(G) + \beta'(G) \leq n(G)$

Let M be the maximum matching set

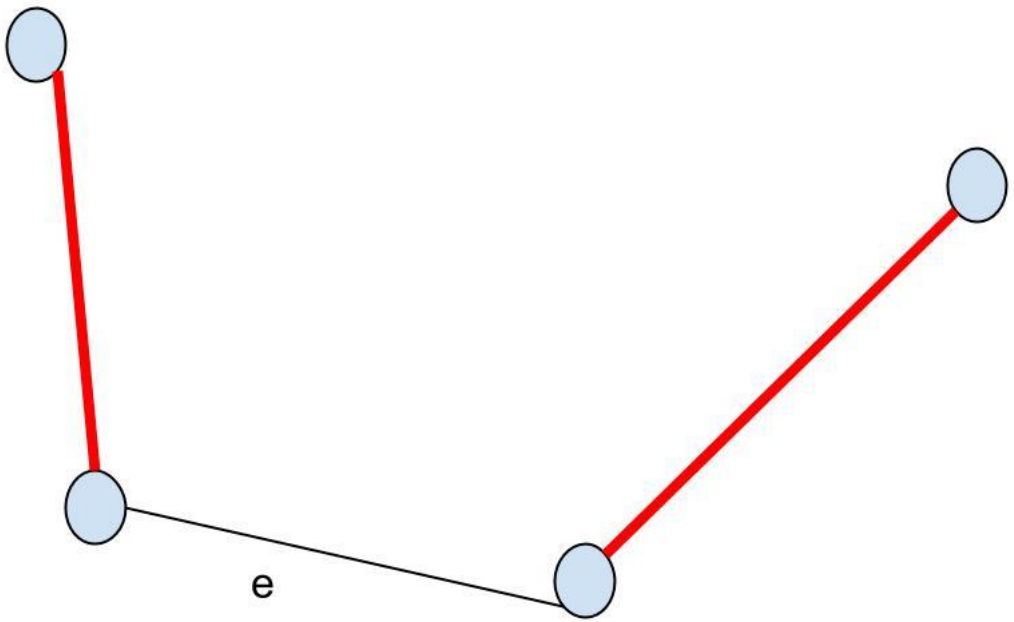



Edge cover M' = M
+ edge incident to
each unsaturated
vertex
unsaturated
vertex
added
edge

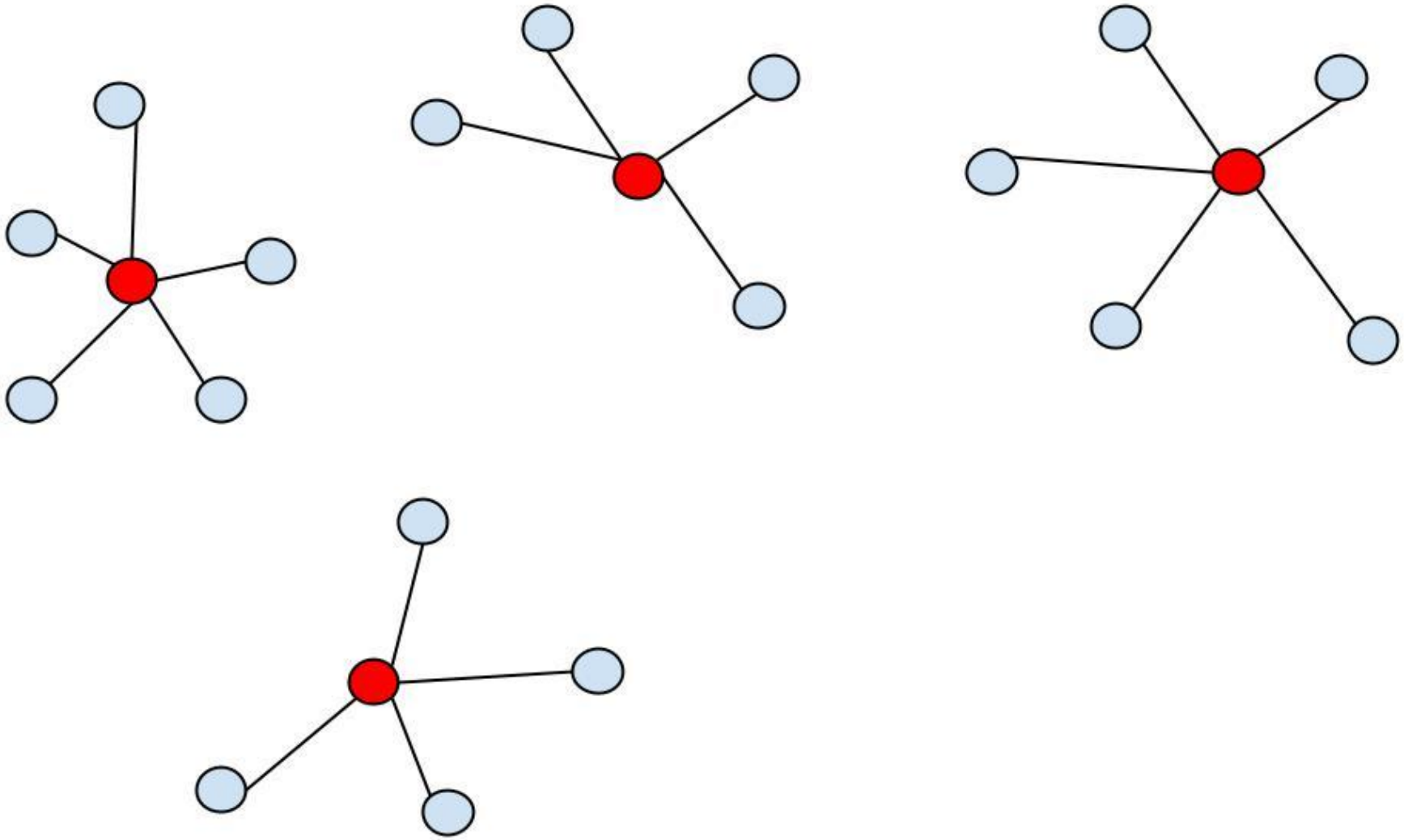
2) Prove $\alpha'(G) + \beta'(G) \geq n(G)$

Let L : minimum edge cover. Suppose L has k connected component

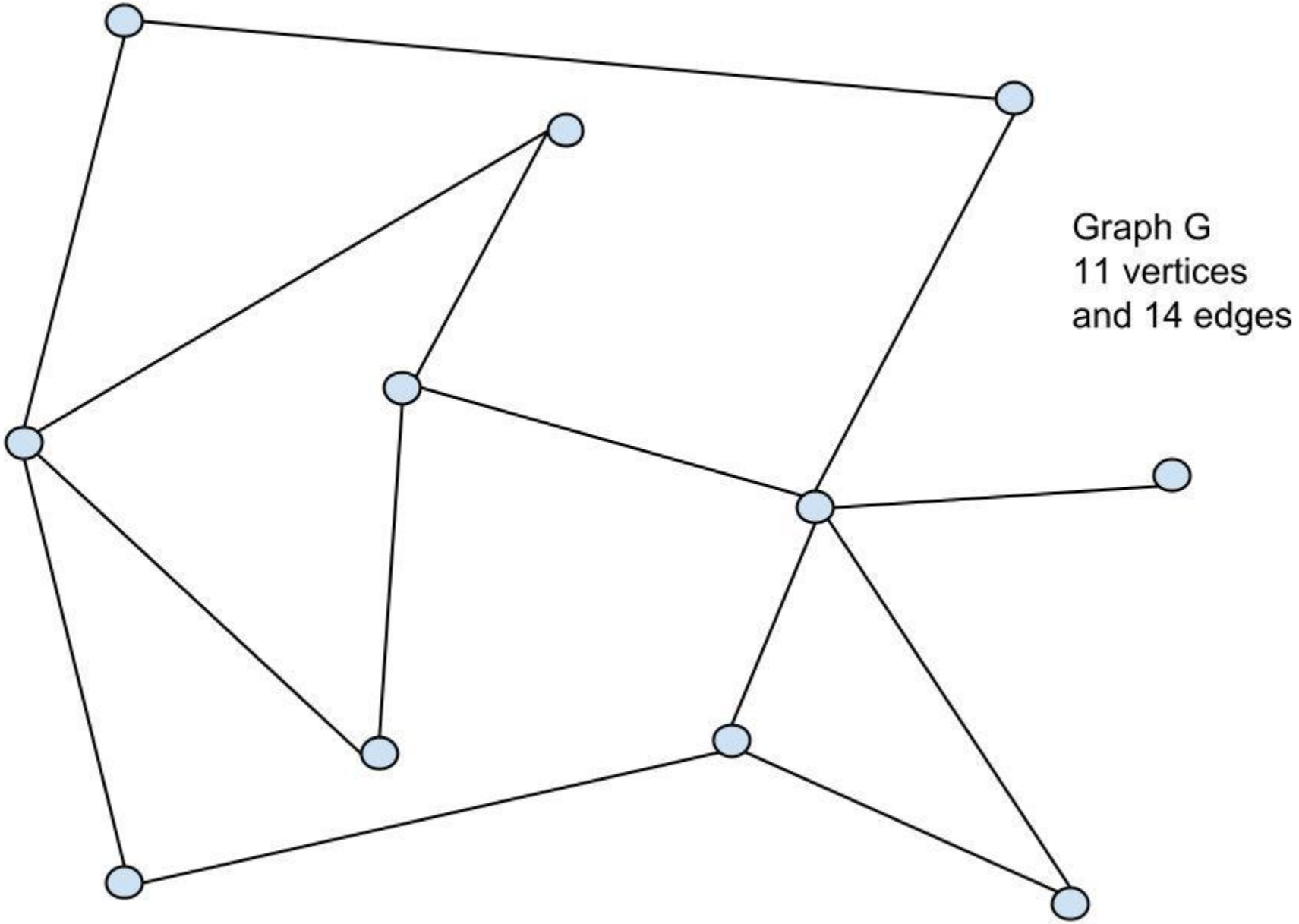
Observation 1



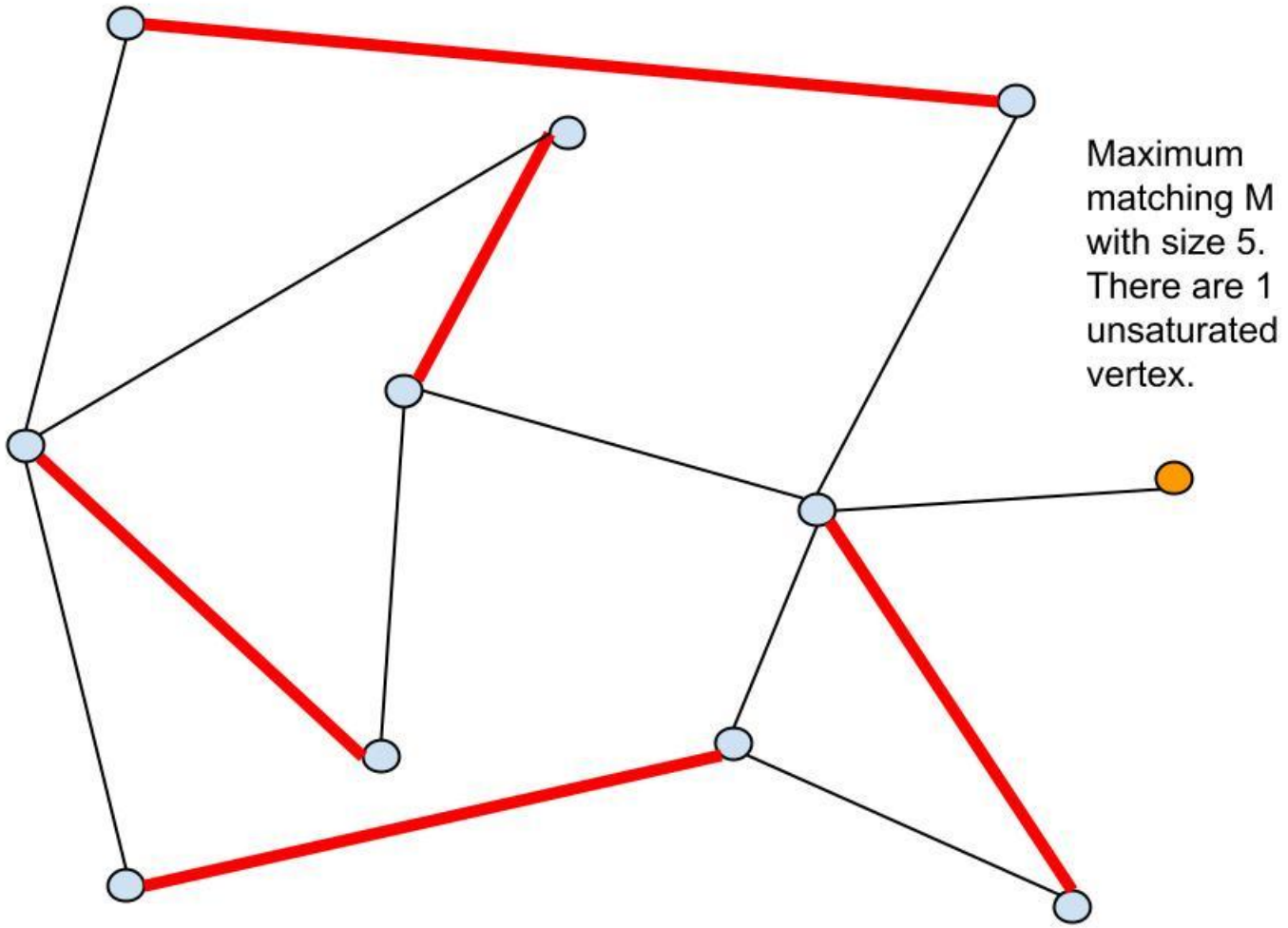
L consists of
k star
components

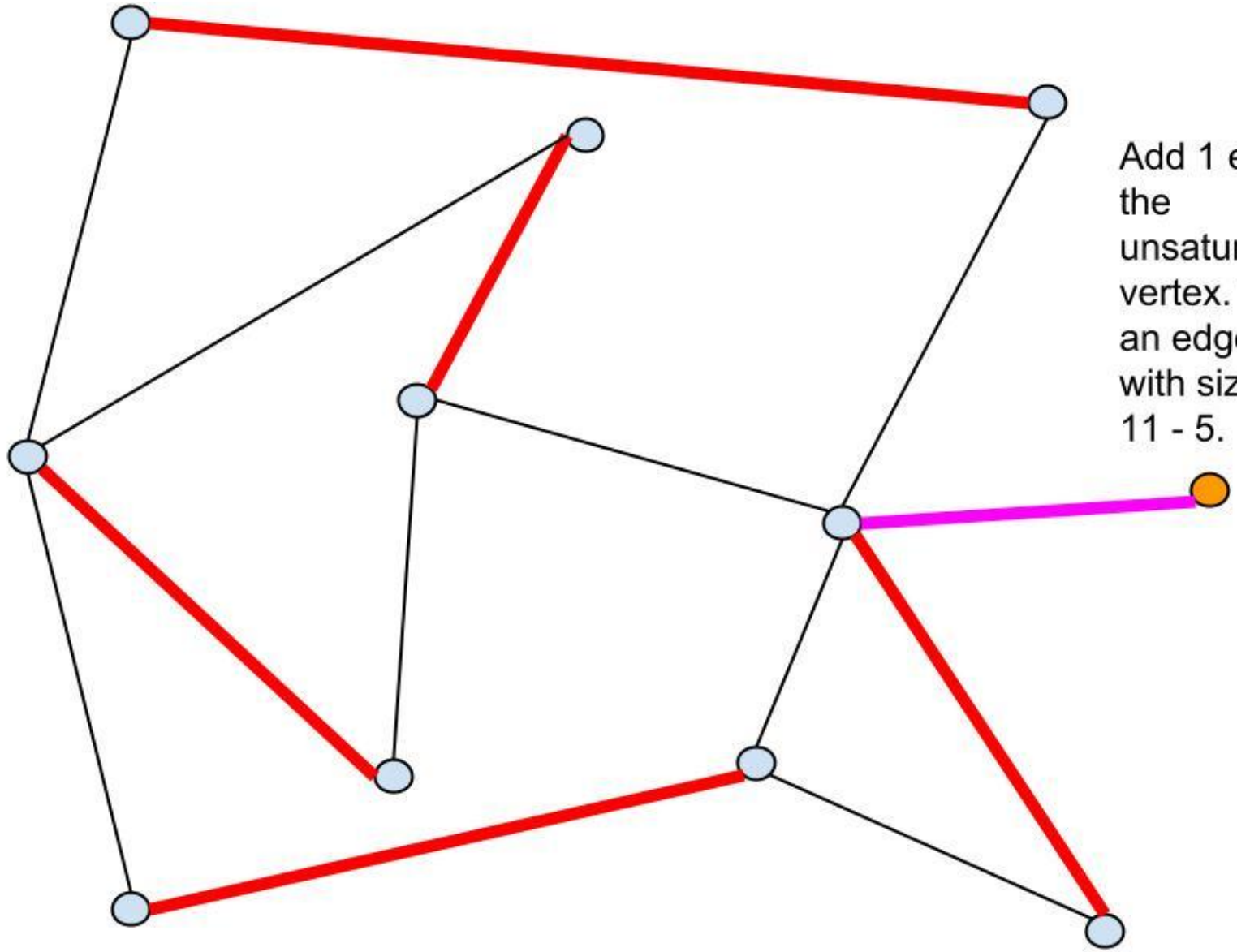


Example



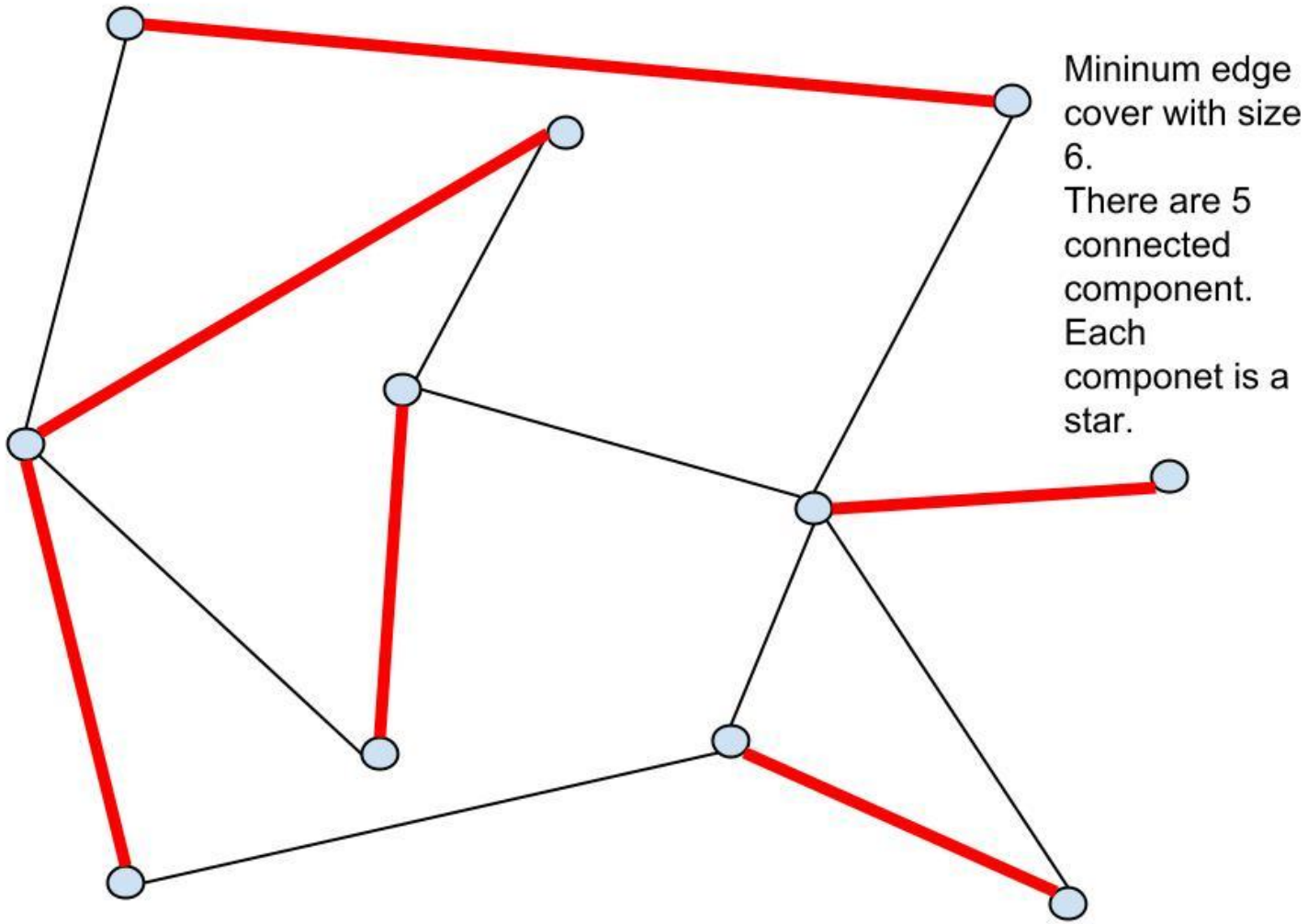
1) Given maximum matching, construct edge cover

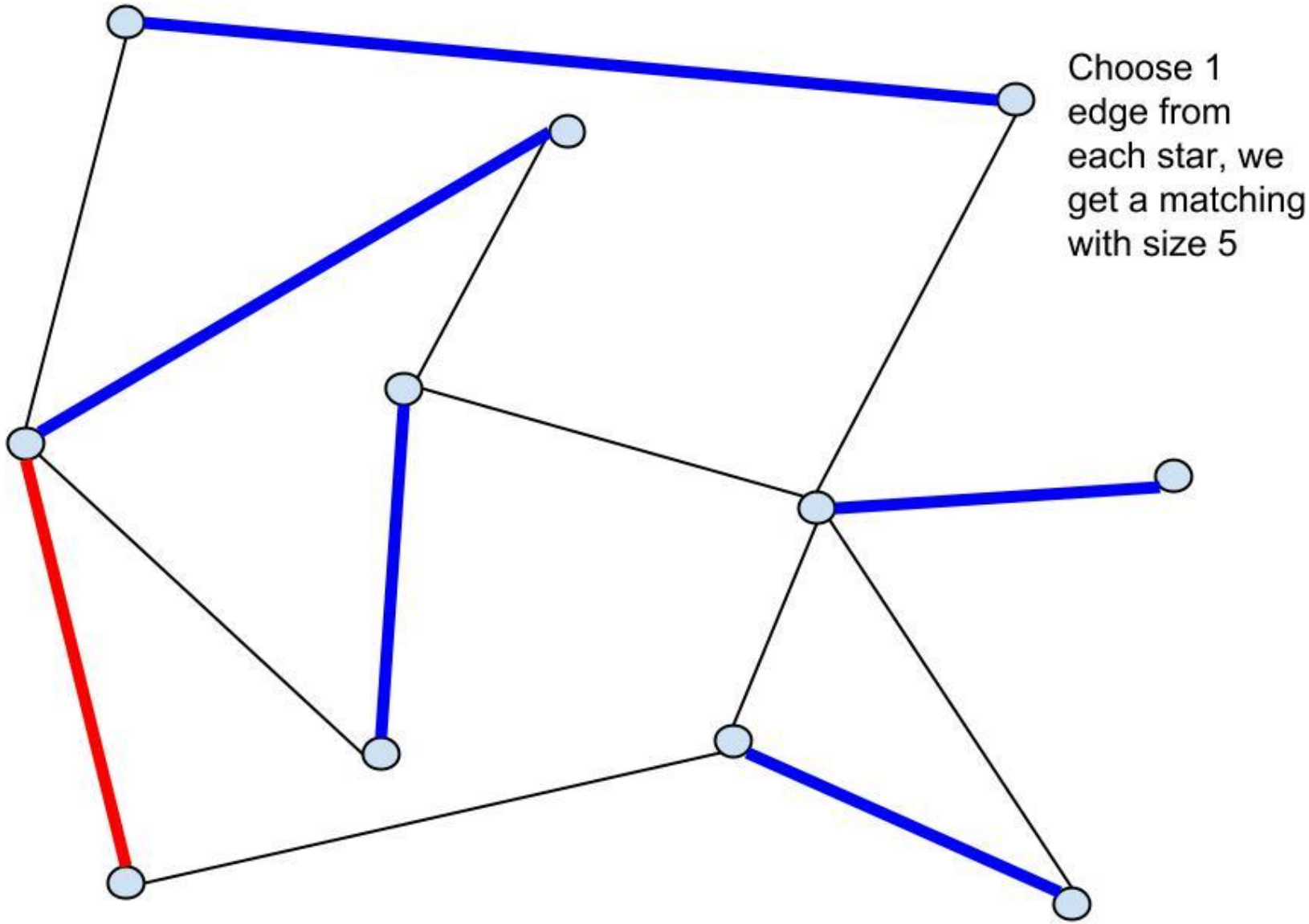




Add 1 edge to the unsaturated vertex. We get an edge cover with size 6 = 11 - 5.

2) Given edge cover set, construct maximum matching set




Choose 1
edge from
each star, we
get a matching
with size 5