Connectivity of $k$-critical graphs

Melanie Foerster

Math 4330: Topics in Graph Theory
Dr. Jeannette Janssen
Lemma (Kainen)

Let $G$ be a graph with $\chi(G) > k$, and let $X, Y$ be a partition of $V(G)$. If $G[X]$ and $G[Y]$ are $k$-colourable, then the edge cut $[X, Y]$ has at least $k$ edges.

Proof. Let $X_1, X_2, \ldots, X_k$ and $Y_1, Y_2, \ldots, Y_k$ be the partitions of $X$ and $Y$ formed by the colour classes in proper $k$-colourings of $G[X]$ and $G[Y]$. If there is no edge between $X_i$ and $Y_j$ then $X_i \cup Y_j$ is an independent set in $G$. We will show that if $|X, Y| < k$ then we can combine colour classes from $G[X]$ and $G[Y]$ in pairs to form a proper $k$-colouring of $G$.

Form a bipartite graph $H$ with vertices $X_1, X_2, \ldots, X_k$ and $Y_1, Y_2, \ldots, Y_k$, putting $X_i Y_j \in E(H)$ if in $G$ there is no edge between the set $X_i$ and the set $Y_j$. If $|X, Y| < k$, then $H$ has more than $k(k - 1)$ edges. Since $m$ vertices can cover at most $km$ edges in a subgraph of $K_k$, then $E(H)$ cannot be covered by $k-1$ vertices. By the Kőnig-Egerváry Theorem, $H$ therefore has a perfect matching $M$.

In $G$, we give colour $i$ to all of $X_i$ and all of the set $Y_j$ to which it is matched by $M$. Since there are no edges joining $X_i$ and $Y_j$, doing this for all $i$ produces a proper $k$-colouring of $G$, which contradicts the hypothesis that $\chi(G) > k$. Hence $|X, Y| \geq k$. □
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Theorem (Dirac, 1953)

Every $k$-critical graph is $k - 1$-edge-connected.
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Every $k$-critical graph is $k-1$-edge-connected.

Proof. Let $G$ be a $k$-critical graph, and let $[X, Y]$ be a minimum edge cut. Since $G$ is $k$-critical, $G[X]$ and $G[Y]$ are $k-1$-colourable.
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Proof. Let $G$ be a $k$-critical graph, and let $[X, Y]$ be a minimum edge cut. Since $G$ is $k$-critical, $G[X]$ and $G[Y]$ are $k - 1$-colourable. Using the lemma with $k - 1$ in place of $k$, we conclude that $|X, Y| \geq k - 1$. □