Theorem (Berge 1957)

A matching M in a graph G is a maximum matching if and only if G has no M-augmenting path.

We have already seen in class that if a matching M is a maximum matching then G has no augmenting path

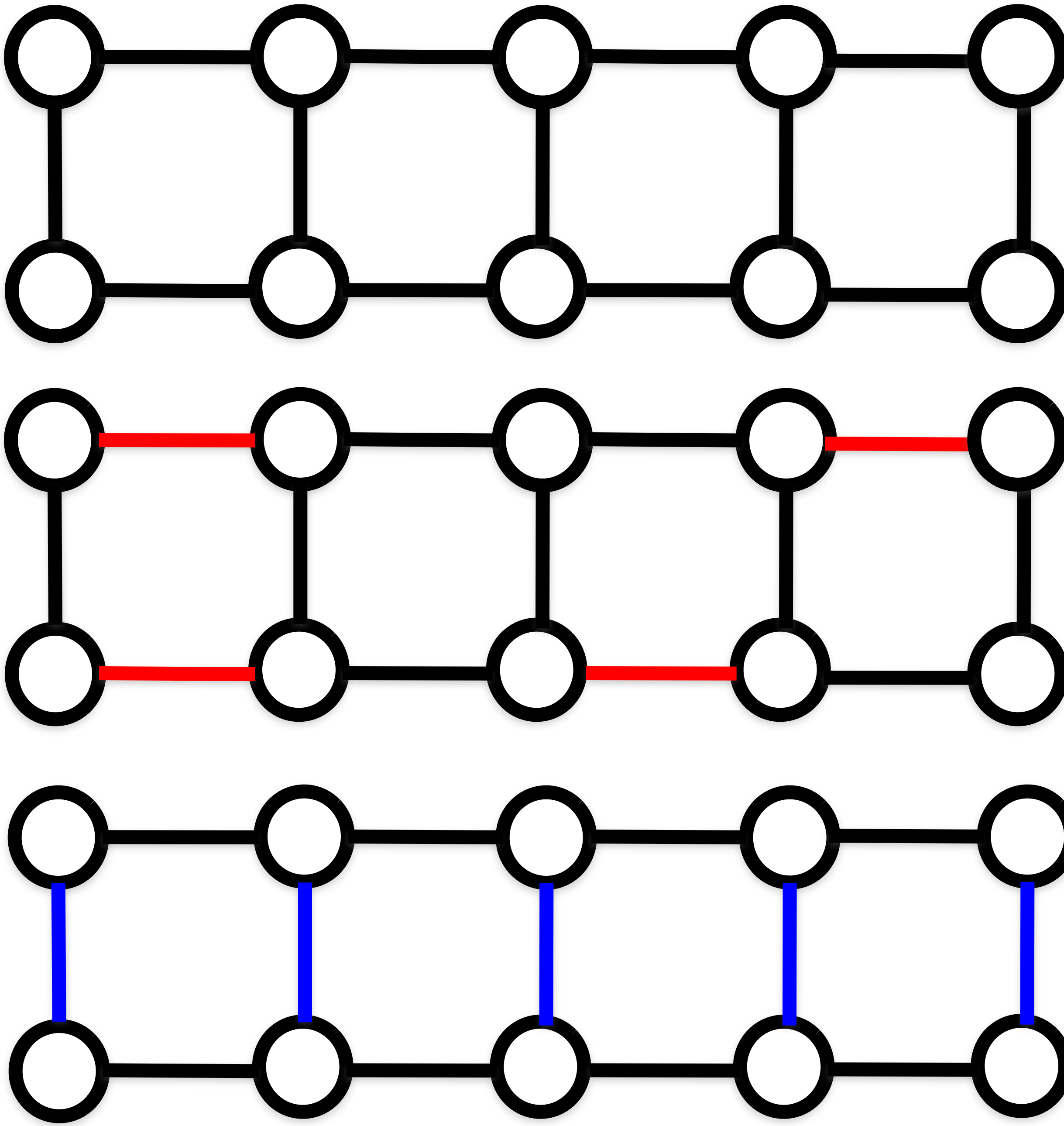
Want to prove that if G has no M-augmenting path then a matching M is a maximum matching

Will prove this by showing that if M is not a maximum matching then G has an M-augmenting path

This is the contrapositive

Suppose M is not a maximum matching.

Then there is a matching M' which is larger than M



G

M

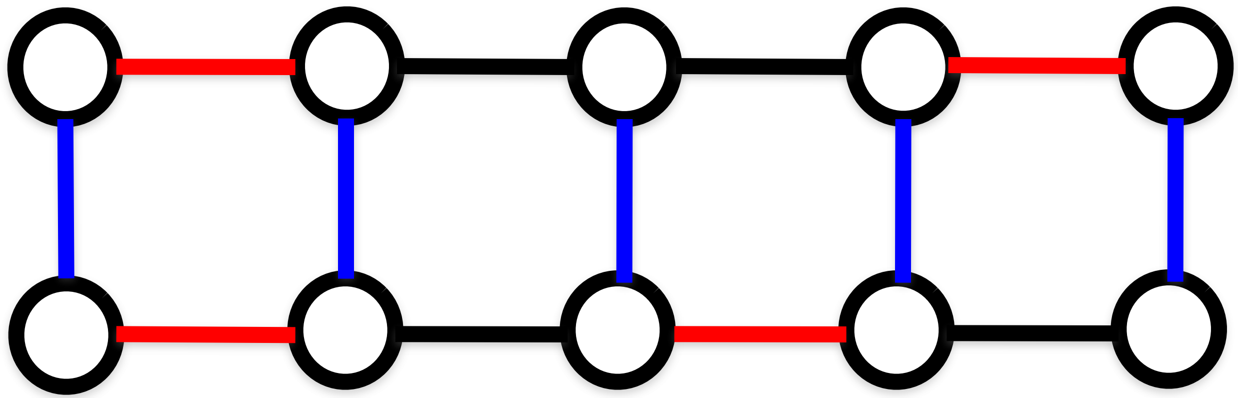
|M |=4

M’

|M’ |=5

Consider the set S of edges that belong to M or M' but not both

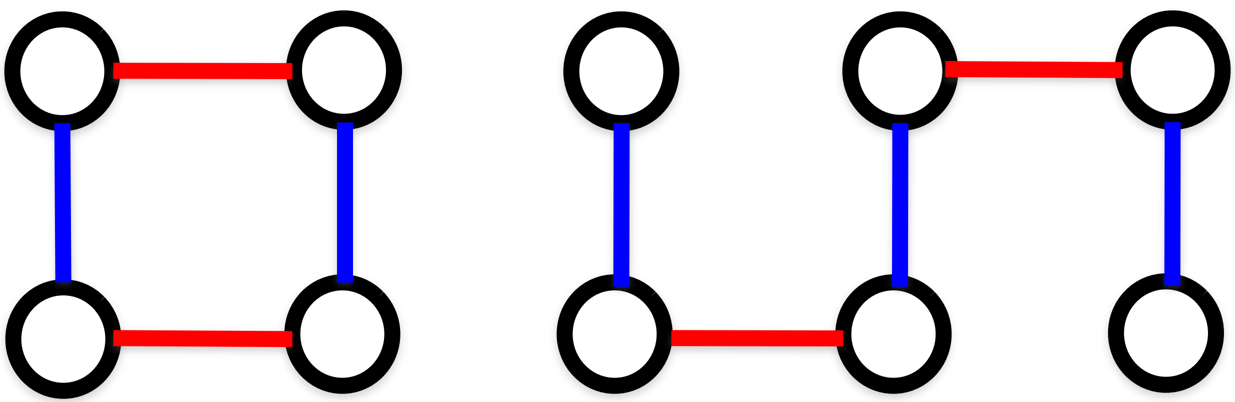
S is called the symmetric difference of M and M'



S

In this example M and M’ share no edges so S is the union of the two matchings

Let H be the graph formed by the edges in S and their endpoints



H

Since both M and M’ are matchings, every vertex of H has degree at most 2

If H had a vertex of degree greater than 2, two edges of M or M’ would have to have a vertex in common which cannot happen in a matching

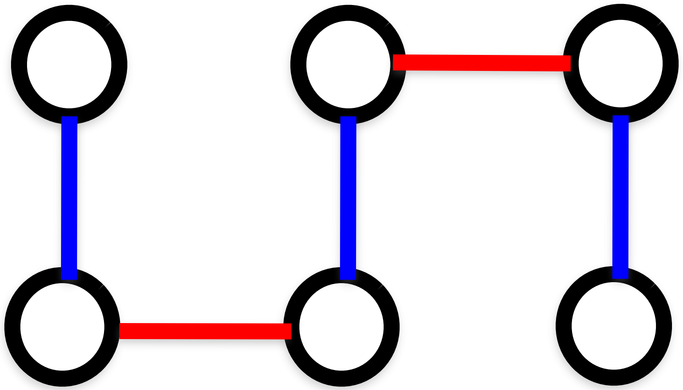
So H is a collection of disjoint paths and cycles

Every cycle and path must be M-alternating so every cycle must be even

If a component is not M-alternating then there would be two adjacent M-edges which cannot happen in a matching

Since M’ is larger than M, H must contain more M’-edges than M-edges

Therefore there must be a component of H that contains more M’-edges than M-edges



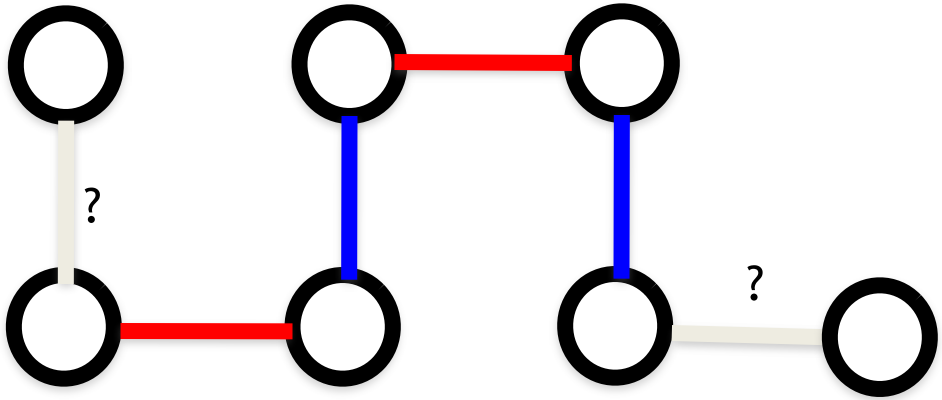
This component must be a path, say P

It cannot be a cycle as all cycles are even so contain the same number of M and M’ edges

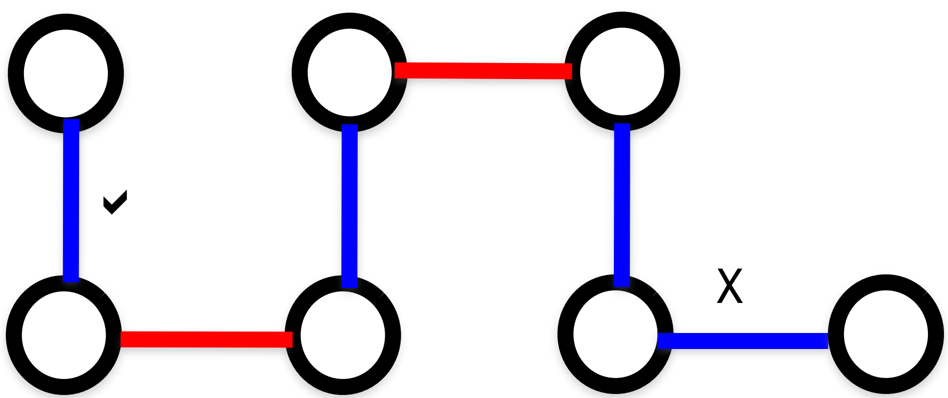
Since P has more M’ edges than M edges P must be of odd length

If P was of even length P would contain the same number of M and M’ edges as it is M-alternating

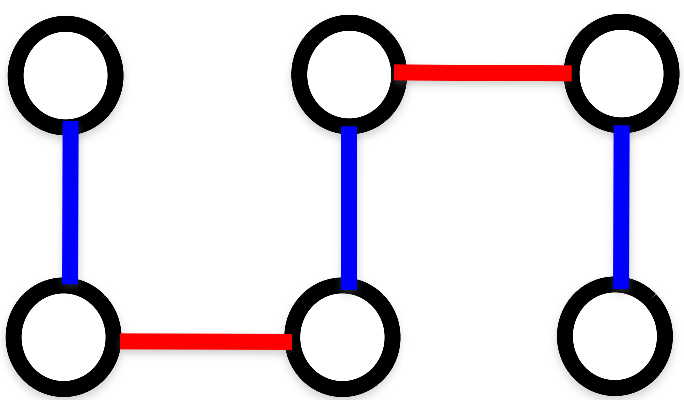
P must also start and end with an M'-edge.



Removing either final edge gives an even alternating path P’



We know that the original path P had more edges in M’ than in M => removed edge must be in M’



We know have that the path must start and finish with an edge from M’

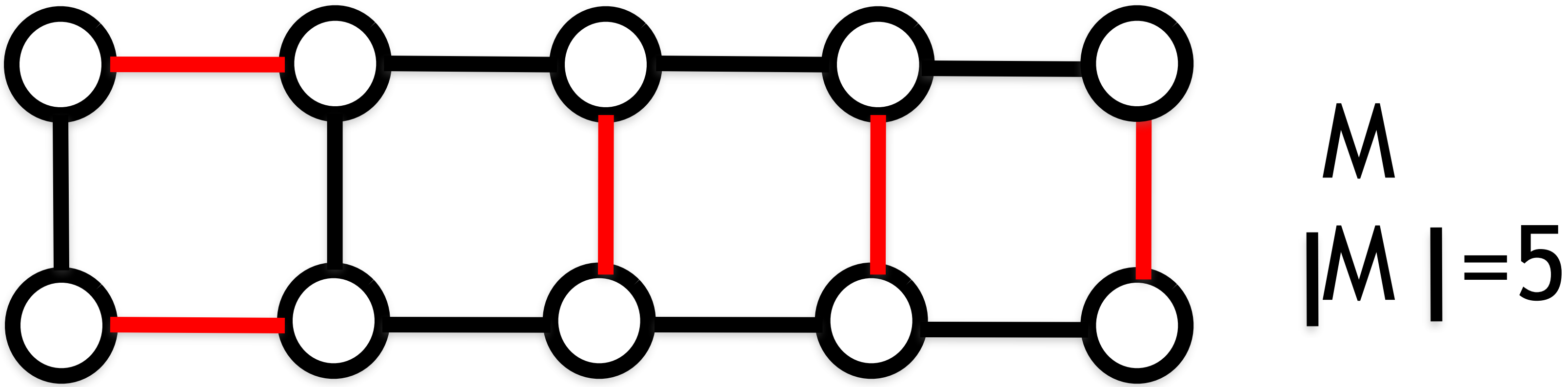
Moreover since P is a maximum connected component of H the endpoints of P must be M-unmatched

If the endpoints were M-matched then the component could be extended so would not be maximum

So P is an M-alternating path whose endpoints are M-unmatched

Thus P is an M-augmenting path

END OF PROOF



In our example the M-augmenting path can be used to extend the matching so it has size 5