MATH 4370/5370
Class Project (Winter term 2015)

To complete the course, students will have to do a class project. The evaluation of the project will be based on a written and revised paper, and a class presentation. Unless there are strong reasons not to, the project may be done in teams of two, and in fact, this is recommended.
The outline of the project is as follows. Early in the term, students will choose a topic from the list given below. The topics given are research-style problems. The team then attempts to solve the problem on their own, without consulting the Web, books, or other sources. If the original problem is solved, the team may go on to solve variations of the original problems.
The solution then must be written up in a strict format that resembles that used in mathematicial publications. A template, in LaTex, will be provided, and must be used. Examples will be made available.
The report/paper must be submitted for review. A first review will be done by your peers, i.e. the other students in the class. The papers will be also reviewed by the instructor. After the review, there will be a chance to make changes.
Finally, the work must be presented to the class in a presentation of 15 minutes.

TOPICS:

1. **Pascal's triangle modulo two.** If the binomial coefficients in Pascal's triangle are written modulo 2, a fractal structure emerges similar to the Sierpinski gadget. Verify this. Find a formula for the exponent of 2 in the prime decomposition of $n!$. Use this to show that, for n a power of 2, all binomial coefficients $\binom{(}{n})k$ for $1 \leq k \leq n$ are even. Use this to prove the fractal structure. Optional: generalize to Pascal's triangle modulo 3 or higher.

2. **Counting latin squares using permanents.** The permanent of an $n \times n$ matrix M is defined as follows:

$$per(M) = \sum_{\sigma \in S_n} M_{i,\pi(i)}.$$

   Here S_n is the set of all permutations of n elements. (This formula is similar to the formal definition of a determinant; for the determinant there is a similar sum, but each term has a plus or minus sign attached). Show that, if M is the incidence matrix of a collection of n subsets of $\{1, 2, \ldots, n\}$, then $per(M)$ is the number of SDRs. Permanents now give another bound on the number of SDRs.

Show that the matrix M with all entries equal to $1/n$ has permanent equal to $n!/n^n$. It is a known result that:

The permanent of an $n \times n$ doubly stochastic matrix M is at least $n!/n^n$, with equality if and only if all entries of M are equal to $1/n$.

Use this result to show the following: If $A_1, \ldots, A_n$ is a collection of subsets of $\{1, \ldots, n\}$ with the property that each element $j \in \{1, \ldots, n\}$ occurs in exactly r of the sets, and each set A_i has size r, then the collection has at least $n!(r/n)^n$ different SDRs. Finally, use this result to find another lower bound on the number of $n \times n$ latin squares. Optional: compare with the lower bound shown in class.

3. **Counting special matchings** Consider vertices labelled $1, 2, \ldots, 2n$ arranged in order around a circle. A non-intersecting matching is a set of n line segments, each connecting two of these vertices, no two segments have the same endpoint, and no two line segments cross. Said another way, it is a matching of that saturates all the vertices, all edges in the matching are straight lines, and no edges cross.

   The Catalan numbers C_n are defined as

   $$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

   It is known (and you may use) that the Catalan numbers count the number of binary words with exactly n zeros and n ones for which every prefix has at least as many ones as zeros. It is also known that the Catalan numbers satisfy the following recurrence:

   $$C_0 = 1,\ C_1 = 1, \quad \sum_{k=1}^{n} C_{k-1}C_{n-k} \text{ for } n \geq 2.$$

   Show that the number of non-intersecting matchings is counted by the Catalan numbers. You may use either the recurrence or use a bijective map to the particular set of binary words counted by the Catalan numbers.

4. **Sets of points and their sums and differences.** Given 25 different positive numbers, prove that you can choose two of them such that none of the other numbers equals either their sum or their difference. Is this still true if the numbers can be zero or negative? Are there other values n for which this is true?

5. **Triangles in a graph.** Let $t_{n,e}$ be the minimum number of triangles can a (simple, undirected) graph on n vertices with e edges can have. Compute $t_{n,e}$

for some small values. Show the following upper bound on $t_{n,e}$:

$$t_{n,e} \geq \frac{e}{3n}(4e - n^2).$$

Give a construction that achieves a value close to the lower bound. Optional: Do the same for the maximum number of triangles.

6. **Counting trees.** Specifically, we count the number of rooted, unlabelled, ordered trees. Let t_n be the number of such trees with exactly n nodes. Note: the fact that the tree is ordered means that these two trees are considered different:

   Consider the map f from the set of all rooted, unlabelled, ordered trees on n nodes to the set of all binary words of length $2n-1$ obtained as follows. Let $v_1, \ldots, v_n$ be a listing of the nodes in pre-order transversal (going counter clockwise around the tree, see also Wikipedia – tree traversals). Then the binary word is formed by replacing each representing each node v_i in the list by a word consisting of d zeros followed by a 1, where d is the number of children of v_i. Show that f indeed gives a word of length $2n-1$. Show that each word in the range of f has exactly n ones and $n-1$ zeros. Show that each word in the range of f has the property that each proper prefix of the word (i.e. each prefix except the word itself) has at least as many zeros than ones. Show that each word in the range of f must start in 0 and end in 1.

   Now let S be the set of all binary words with n zeros and $n-1$ ones, and ending in 1. What is the size of S? Consider this to be the co-domain of f. Show that f is not onto.

   Now, for any word w in S, let $t(w)$ be the number of proper prefixes of w that have more ones than zeros. Consider the map $g : S \to S$ defined as follows: for $w \in S$, let u be the shortest prefix of w with ones than zeros, and let $w = uz$. Then $g(w) = zu$. For example, $g(01101) = 01011$, and $u = 011$, $z = 01$. Show that, if w is not in the range of g, then $t(g(w)) = t(w) - 1$. g reduces the number of proper prefixes that have more ones than zeros. Show also that g is an injection. Conclude that the range of f has size $\frac{1}{n}|S|$. Use this to count the number of rooted, unlabelled, ordered trees on n nodes.