MATH 4370/5370
Material for self-study, Jan. 27-Feb. 5

A partially ordered set, or poset, is a set P with a binary relation $\prec$ which is reflexive, anti-symmetric and transitive. Two elements $a, b \in P$ are comparable if $a \prec b$ or $b \prec a$, otherwise they are incomparable. A chain is a subset C of P so that any two elements of C are comparable. A chain C is maximal if there is no other chain that contains C as a proper subset. An antichain is a subset A of P so that any two elements of A are comparable. An antichain A is maximal if there is no other antichain that contains A as a proper subset.
A maximal element of P is an element a so that, for any $b \in P$, $a \prec b \Rightarrow a = b$. A minimal element is defined similarly.
Dilworth Let P be a poset. The minimum number m of disjoint chains which together contain all elements of P is equal to the maximum number M of elements in an antichain of P.
Since an antichain and a chain can intersect in at most one element, we have that $m \leq M$. To prove the other part, use induction on $|P|$. If $|P| = 0$ there is nothing to prove. Let C be a maximal chain in P. If every antichain in $P \setminus C$ contains at most $M - 1$ elements, we are done. So assume that $\{\alpha_1, \ldots, \alpha_M\}$ is an antichain in $P \setminus C$. Define $S^- = \{x \in P : \exists i, x \prec \alpha_i\}$, and define S^+ analogously. Since C is maximal, the largest element in C is not in S^- and hence $|S^-| < |P|$ and by the induction hypothesis, the theorem holds for S^-. Hence S^- is the union of M disjoint chains. Moreover, each of these chains has exactly one of the elements α_i as its maximal element. Similarly, S^+ is the union of M disjoint chains, each of which has exactly one of the elements α_i as its minimal element. Combining the chains in S^- and S^+ that contain the same α_i, we obtain M disjoint chains whose union is P.
[Minsky] Let P be a partially ordered set. If P possesses no chain of $m+1$ elements, then P is the union of m antichains.   Induction on m. If $m = 1$, then all elements of P are incomparable, and P is itself an antichain. Let $m \geq 2$ and assume the theorem is true for $m - 1$. Let M be the set of maximal elements of P. Clearly, M is an antichain. Let C be any maximal chain in P. Then C must contain an element of M. Therefore, $P \setminus M$ possesses no chain of m elements. By the induction hypothesis, $P \setminus M$ is the union of $m - 1$ antichains. This proves the theorem.
[Sperner's theorem] If $A_1, \ldots, A_m$ are subsets of $[n]$ so that no two sets A_i are subsets of one another, then $m \leq \binom{n}{\lfloor n/2 \rfloor}$.
To prove this, consider the poset of subsets of $[n]$ and the relation $\subseteq$. See Jukna, Theorem 8.3.
Such a collection of sets is sometimes called an intersecting family. See Sections 7.1 and 7.2, Jukna.

The following is a folklore result. If the edges of the complete graph K_7 are coloured red and blue, then there must be a red or a blue triangle. In general, the Ramsey number $R(r, k)$ is the smallest integer n so that, if the edges of K_n are coloured with r colours, there is a always a monochromatic K_k. Ramsey's theorem says that this number is well-defined, i.e. there always exists such an integer n. Read more about Ramsely numbers in Cameron, Section 10.1–4.