A partially ordered set, or poset, is a set $P$ with a binary relation $\prec$ which is reflexive, anti-symmetric and transitive. Two elements $a, b \in P$ are comparable if $a \prec b$ or $b \prec a$, otherwise they are incomparable. A chain is a subset $C$ of $P$ so that any two elements of $C$ are comparable. A chain $C$ is maximal if there is no other chain that contains $C$ as a proper subset. An antichain is a subset $A$ of $P$ so that any two elements of $A$ are comparable. An antichain $A$ is maximal if there is no other antichain that contains $A$ as a proper subset.

A maximal element of $P$ is an element $a$ so that, for any $b \in P$, $a \prec b \Rightarrow a = b$. A minimal element is defined similarly.

Dilworth Let $P$ be a poset. The minimum number $m$ of disjoint chains which together contain all elements of $P$ is equal to the maximum number $M$ of elements in an antichain of $P$.

Since an antichain and a chain can intersect in at most one element, we have that $m \leq M$. To prove the other part, use induction on $|P|$. If $|P| = 0$ there is nothing to prove. Let $C$ be a maximal chain in $P$. If every antichain in $P \setminus C$ contains at most $M - 1$ elements, we are done. So assume that $\{\alpha_1, \ldots, \alpha_M\}$ is an antichain in $P \setminus C$. Define $S^- = \{x \in P : \exists i, x \prec \alpha_i\}$, and define $S^+$ analogously. Since $C$ is maximal, the largest element in $C$ is not in $S^-$ and hence $|S^-| < |P|$ and by the induction hypothesis, the theorem holds for $S^-$. Hence $S^-$ is the union of $M$ disjoint chains. Moreover, each of these chains has exactly one of the elements $\alpha_i$ as its maximal element. Similarly, $S^+$ is the union of $M$ disjoint chains, each of which has exactly one of the elements $\alpha_i$ as its minimal element. Combining the chains in $S^-$ and $S^+$ that contain the same $\alpha_i$, we obtain $M$ disjoint chains whose union is $P$.

[Sperner’s theorem] If $A_1, \ldots, A_m$ are subsets of $\left[\frac{n}{2}\right]$ so that no two sets $A_i$ are subsets of one another, then $m \leq \left(\frac{n}{\left(\frac{n}{2}\right)}\right)$. To prove this, consider the poset of subsets of $\left[\frac{n}{2}\right]$ and the relation $\subseteq$. See Jukna, Theorem 8.3.

Such a collection of sets is sometimes called an intersecting family. See Sections 7.1 and 7.2, Jukna.
The following is a folklore result. If the edges of the complete graph $K_7$ are coloured red and blue, then there must be a red or a blue triangle. In general, the Ramsey number $R(r, k)$ is the smallest integer $n$ so that, if the edges of $K_n$ are coloured with $r$ colours, there is always a monochromatic $K_k$. Ramsey’s theorem says that this number is well-defined, i.e. there always exists such an integer $n$. Read more about Ramsely numbers in Cameron, Section 10.1–4.