Combinatorics – Problem set 1

Due Thursday, Jan. 15, beginning of class

Write a careful argument to prove your answer. Be neat and precise. Try to make your argument as short as possible, but no shorter. Unmarked problems should be done by all students. Problems marked with a course number (e.g. MATH 4370) should be done by students enrolled in that course only.

1. Let $0 < r \leq n$. Suppose that $A_1, \ldots, A_n$ is a collection of sets so that $|A_i| \geq r$ for all $1 \leq i \leq n$. Assume that HC holds. Show that there are at least $r!$ SDRs. *Proof sketch done in class.*

2. (a) Let $A_1, \ldots, A_n$ be a collection of sets with the property that every set has size at least r, and each element occurs in at most r sets of the collection. Show that HC holds for the collection. (b) Alice challenges Bob to a game. The rules are as follows. Bob gets to divide a deck of cards into 13 stacks of four cards each. Alice then must choose one card from each deck so that she has a full house, i.e. all 13 cards representing each one of Ace, $2, \ldots, 10$, Jack, Queen, King. Alice wins if she succeeds, Bob wins if she fails. Bob declines the challenge, saying that he cannot possibly win. Is Bob right? Give a winning strategy for Bob, or show that no such strategy exists.

3. (a) Give two 2×4 latin rectangles so that the first rectangle has more completions to a latin square than the second rectangle. Explain your answer.
(b) How many ways possibilities are there for the next row of the following latin rectangle:

$$\begin{array}{ccccccc} 4 & 2 & 3 & 5 & 6 & 1 & 7 \\ 5 & 3 & 2 & 7 & 1 & 6 & 4 \\ 6 & 7 & 4 & 1 & 3 & 5 & 2 \\ 7 & 6 & 5 & 3 & 4 & 2 & 1 \end{array}$$

Give the collection of sets so that the next row of the rectangle is an SDR of this collection.

4. An $n \times n$ latin square can also be seen as a collection of n^2 triples from $S \times S \times S$, where $S = \{1, \ldots, n\}$. The triples give the row, column

and entry of the n^2 subsquares of the latin square. For example, the collection $\{(1,1,1),(1,2,2),(2,1,2),(2,2,1)\}$ corresponds to the latin square $\begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix}$

(a) Show that a collection of n^2 triples is a latin square if and only if every pair of triples differs in at least two coordinates.

(b) Given (a), it is clear that if we permute the coordinates of all triples in the same way, then the resulting set of triples again corresponds to a latin square. For example, if the first and second coordinate are switched, then the resulting latin square will be the transpose of the original (rows and columns switched. Demonstrate with the latin square below what latin square results if the second and third coordinate are switched. Describe this operation without mentioning the triples.

1	2	3	4	5	6
2	4	1	5	6	3
3	1	6	2	4	5
4	3	5	6	1	2
5	6	2	1	3	4
6	5	4	3	2	1

5. [MATH5370] Let $A_1, \ldots, A_n$ be a collection of subsets of $\{1, 2, \ldots, n\}$. Let M be the incidence matrix of the collection. Precisely, let $M_{ij} = 1$ if $i \in A_j$, and 0 otherwise. Show that, if M is invertible (non-singular), then the collection of sets has an SDR.

6. [MATH 4370] Prove the following generalization of Hall's theorem: Given a collection of subsets $A_1, \ldots, A_n$. If, for all $J \subseteq \{1, 2, \ldots, n\}$, $|A(J)| \geq |J| - r$, then there exists a sub-collection of $n - r$ of these sets which have an SDR. *Hint: add r extra elements to all sets.*