Combinatorics – Problem set 2

Due Thursday, Jan. 22, beginning of class

Write a careful argument to prove your answer. Be neat and precise. Try to make your argument as short as possible, but no shorter. Unmarked problems should be done by all students. Problems marked with a course number (e.g. MATH 4370) should be done by students enrolled in that course only.

1. Show that, for all integers $n \geq 1$,
$$n! \leq ne\left(\frac{n}{e}\right)^n.$$

2. Stirling's formula says that $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. Prove or disprove the following statements. Use precise mathematical arguments, and use the definition of the Big-Oh symbols involved. (*For a brief tutorial on Big-Oh notation, see the wiki page, or follow the link on the course Web page.*

   (a) $n!$ is $O(n^{n+\frac{1}{2}})$,

   (b) $n!$ is $\theta(n^{n+\frac{1}{2}}e^{-n})$,

   (c) $n!$ is $o(n^n)$.

3. For all integers n, the *Euler totient function* $\phi : \mathbb{N} \to \mathbb{N}$ is defined as follows: $\phi(n)$ is the number of integers in $\{1, \ldots, n\}$ which are relatively prime to n. (Two integers a, b are relatively prime if $gcd(a, b) = 1$, so their only common divisor is 1.) For example, $\phi(6) = 2$, because only 1 and 5 are relatively prime to 6.

   (a) Use inclusion/exclusion to compute $\phi(300)$.

   (b) [4370] Let $n = p_1^{\alpha_1}p_2^{\alpha_2}$. Show that $\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})$.

   (c) [5370]) Show that, for all integers $n \geq 1$ with prime decomposition $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$,
$$\phi(n) = n\left(1 - \frac{1}{p_1}\right) \ldots \left(1 - \frac{1}{p_k}\right).$$
   Use this formula to compute $\phi(300)$, and verify that the answer is the same as in (a).