Combinatorics – Problem set 3

Due Thursday, Jan. 29, beginning of class

Write a careful argument to prove your answer. **Be precise**. *Define every induction argument formally, and state the induction hypothesis explicitly. Make your argument as short as possible, but no shorter. Unmarked problems should be done by all students. Problems marked with a course number (e.g. MATH 4370) should be done by students enrolled in that course only.*

1. (a) Use the methods shown in class to construct two orthogonal latin squares of order 12. (b) Give a lower bound on the number of MOLS of the following orders: 15, 19, 20, 64. Indicate how such MOLS could be constructed.

2. A finite field of order 8 can be constructed using the method shown in class. Precisely, the elements of the field are the polynomials of degree less than 3 over ${}_2$, modulo an irreducible polynomial. Use the irreducible polynomial $1 + x^2 + x^3$ to construct a finite field of order 8. First, give the addition and multiplication table. Then, using the field, give three MOLS of order 8. Is this the maximal number of MOLS of order 8?

3. Use inclusion/exclusion to find a formula for the number of surjections from $[n]$ to $[k]$, $k \leq n$. (The notation $[n]$ is used to denote the set $\{1, 2, \ldots, n\}$.)

4. Let d_n be the number of derangements of $[n]$. (a) Compute d_1, d_2, d_3, and d_4. (b) Show that, if n is even, then d_n is odd. (c) What about n odd?

5. Let S_n be the set of all permutations of $[n]$. An *inversion* in a permutation $\sigma \in S_n$ is a pair (i, j), $i, j \in [n]$, so that $i < j$ and $\sigma(i) > \sigma(j)$. The number of inversions of σ is denoted $inv(\sigma)$. A permutation is called *even* if it has an even number of inversions, and *odd* otherwise. (a) Give a permutation of [7] with at least 4 inversions, and determine whether it is odd or even.

   (b) The formal definition of the determinant of an $n \times n$ matrix A is as follows.

$$det(A) = \sum_{\sigma \in S_n} (-1)^{inv(\sigma)} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

Let A be so that all entries on the diagonal are zero, and all entries off the diagonal are in $\{-1, 1\}$. Show that, if n is even, $det(A) = 0$. *Hint: use the previous question.*

6. [4370] The four walls and ceiling of a room are to be painted with five colors available. (Colours may be used more than once, or not at all). How many ways can this be done if bordering sides of the room must have different colours?

7. [5370] How many ways are there to seat n couples around a table so that no couple sits next to each other?