Combinatorics – Problem set 4

Due Thursday, Feb. 12, beginning of class

Write a careful argument to prove your answer. **Be precise**. *Define every induction argument formally, and state the induction hypothesis explicitly. Make your argument as short as possible, but no shorter. Unmarked problems should be done by all students. Problems marked with a course number (e.g. MATH 4370) should be done by students enrolled in that course only.*

1. Consider Füredi's theorem which gives a lower bound on the size of a maximal intersecting family (presented in class Feb. 4, also see Jukna, Theorem 8.4. The bound obtained from the counting argument is that

$$|\mathcal{F}| \geq \frac{\binom{n}{k}}{1 + \binom{n-1}{k-1}}.$$

   (a) This bound works for all values of k and n, not just for $n \leq k^2/2 \log k$. Show, by giving some examples, that the bound is not useful for small values of k.

   (b) Fill in the details of the argument that the lower bound, with the condition that $n \leq k^2/(1 + 2 \log k)$, implies that $|\mathcal{F}| > k^2$. (My derivation in class had one small error in the first step; see also Jukna's proof).

2. Give lower and upper bounds for the Ramsey numbers $R(4, 3)$ and $R(4, 4)$. You can use the bound shown in class, but you may improve on it by giving your own argument. Note that $R(5, 5)$ is still unknown!

3. Consider the proof of Dilworth' theorem given in the notes. Give a detailed proof of the following facts, used in the proof.

   (a) Every maximal chain in a poset P contains a maximal and a minimal element of P. (See notes for definitions.)

   (b) Explain the sentence "If every antichain in $P \setminus C$ contains at most $M - 1$ elements, we are done."

   (c) Give the argument why each of the M chains in S^- contains exactly one of the elements α_i as maximal element

4. Given a sequence of distinct real numbers $\{a_i\}_{i=1}^n$. Define the following partial order on $[n]$: $i \prec j$ precisely when $i = j$, or $i < j$ and $a_i < a_j$.

   (a) Show that $\prec$ is a partial order.

   (b) Show that a chain corresponds to an increasing subsequence. What does an antichain correspond to? Justify you answer.

   (c) Use Dilworth' theorem to show that, if $n = rs + 1$, then any sequence of n distinct numbers either has an increasing subsequence of length r, or a decreasing sequence of length s. (We saw a direct proof of this in class.)

5. Show that the bound in the Erdős-Szekeres is best possible. Precisely, give a sequence of $n = rs$ distinct numbers which does not have an increasing sequence of length r or a decreasing sequence of length s.

6. Show that every set of $n + 1$ elements chosen from the set $[2n]$ must contain a pair of integers whose sum is $2n + 1$.

7. [4370] Given a poset P, an *upper bound* of two elements $a, b \in P$ is an element c so that $a \prec c$ and $b \prec c$. This element c is a *least upper bound* if, for any other upper bound d of a and b, $c \prec d$. Similarly, we can define lower bound and greatest lower bound. A *lattice* is a poset where each pair of elements has a greatest lower bound and least upper bound.

   (a) Show that, if a pair of elements has a least upper bound, then it is *unique*.

   (b) Show that a lattice P has a unique *greatest* element, i.e. an element M so that $a \prec M$ for all $a \in P$.

8. [5370] Prove the following: if $n > srp$, then any sequence of n real numbers must contain either a strictly increasing sequence of length s, a strictly decreasing sequence of length r, or a constant subsequence of length p.