Combinatorics – Problem set 5

Due Thursday, Feb. 26, beginning of class

Write a careful argument to prove your answer. **Be precise**. *Define every induction argument formally, and state the induction hypothesis explicitly. Make your argument as short as possible, but no shorter. Unmarked problems should be done by all students. Problems marked with a course number (e.g. MATH 4370) should be done by students enrolled in that course only.*

1. Find the generating function of the sequence defined by $a_n = n^2$.

2. Assume that $a(x)$ is the generating function of the sequence $\{a_n\}_{n\geq 0}$. Give the generating functions of the following sequences:

   (a) $1, 2, 3, a_3, a_4, a_5, \ldots$

   (b) $a_0, -a_1, a_2, -a_3, a_4, \ldots$

   (c) $\{a_n + 2a_{n+2}\}_{n\geq 0}$

3. The n-th subdivision of an equilateral triangle is the division of the triangle into equally sized triangles that are $1/n$-th the size. For example, below is the third subdivision. (The 0-th subdivision is the triangle itself.)

   (a) [4370] Let t_n be the number of triangles of smallest size in the n-th subdivision. So $t_0 = 1$, $t_1 = 4$. Derive a recurrence for t_n. Find the generating function. Use this to find a closed formula.

   (b) [5370] Let p_n be the number of parallellograms consisting of two small triangles in the n-th subdivision. So $p_1 = 0$, $p_1 = 0$, $p_2 = 3$. Derive a recurrence for p_n. (*Hint: consider separately the parallellograms that have a vertex on the bottom of the triangle, and those that don't.*) Find the generating function. Use this to find a closed formula.