Combinatorics – Problem set 6

Due Thursday, March 12, beginning of class

Write a careful argument to prove your answer. **Be precise**. *Make your argument as short as possible, but no shorter. Unmarked problems should be done by all students.*

1. Calculate the first three coefficients of the reciprocal of the power series

$$p(x) = 1 + x^2 + x^3 + x^5 + x^7 + x^{11} + \dots$$

Remember that the reciprocal of the power series $p(x)$ is the power series $b(x)$ so that $p(x)b(x) = 1$

2. Calculate the first three coefficients of the inverse of the power series

$$a(x) = \sum_{n\geq 1} \binom{2n}{n} x^n = 2x + 6x^2 + 20x^3 + 70x^4 + \dots$$

Remember that the inverse of a power series $a(x)$ is the power series $b(x)$ so that $a(b(x)) = b(a(x)) = 1$.

3. Use generating functions to show *Vandermonde's identity*:

$$\binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k}$$

4. Suppose a candy store has unlimited supplies of three types of candy. A costs 1 cent, B costs 2 cent, and C costs 5 cents. For $n \geq 0$, let a_n be the number of ways that n cents can be spent buying these three candies. For example, 5 cents can be spent buying one of C, or one of B and three of A, or two of B and one of A, or five of A. So $a_5 = 4$. In class, we saw that the generating function of $\{a_n\}_{n\geq 0}$ equals

$$a(x) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^5}\right).$$

   (a) Let b_n be the number of ways to spend n cents so that the selection includes every type of candy at least once. For example, $b_5 = 0$, and $b_8 = 1$. What is the generating function of $\{b_n\}_{n\geq 0}$?

(b) Let c_n be the number of ways to spend n cents so that each selection includes *at most* 4 candies of type A. So $c_5 = 3$. What is the generating function of $\{c_n\}_{n \geq 0}$?

5. Recall that e^x is defined to be the formal power series $e^x = \sum_{n \geq 0} \frac{x^n}{n!}$.. Therefore, we have that

$$e^{2x} = \sum_{n \geq 0} \frac{(2x)^n}{n!} = \sum_{n \geq 0} \frac{2^n x^n}{n!}.$$

Show that $e^{2x} = (e^x)^2 = (e^x)(e^x)$ by comparing the coefficients in front of x^n for the left hand side and right hand side.