December 2008 Solutions

1) Consider the first $2n$ positive integers $1, 2, 3, \ldots, 2n-1, 2n$. Pair off the integers as follows:

$$(1, 2n), (2, 2n-1), (3, 2n-2), \ldots, (n, n+1),$$

and take the product of each of the pairs. Show that each product pair is unique, i.e. not equal to any other product pair.

Proof. Note that each pair is of the form $(m, 2n-(m-1))$ for $m \in \{1, 2, \cdots, n\}$. Consider the difference of consecutive terms $(k, 2n-(k-1))$ and $(k+1, 2n-k)$.

$$\begin{aligned}(k+1)(2n-k) - k(2n-k+1) &= 2nk - k^2 + 2n - k - 2nk + k^2 - k \\ &= 2n - 2k \\ &= 2(n-k)\end{aligned}$$

Since $k+1 \leq n$, have $k < n$, so $2(n-k) > 0$. That is, each product pair is strictly larger than the product pair which preceeded it. That is, the product pairs are strictly increasing. As such, no two can be equal. □

2) Geoff and Meghan play the following game with a heap of n tokens: A move is to choose a heap of size at least 2 (for the first move, this is only one heap to chose) and split it into two non-empty heaps. The game ends when all the heaps have size 1 and the player who split the last heap is the winner (equivalently, if on a player's move, all the heaps are of size 1, then he/she is the loser and the other player is the winner). If Geoff moves first, determine, for all positive integers n, whether Geoff or Meghan wins the game.

Proof. For any n, there are exactly $n-1$ moves. To see this, think of the heap as a row of n dots. On each move, a player is drawing a line between two dots (i.e. splitting a heap into two heaps). There are $n-1$ such lines that can be drawn before there are no more available lines. Thus, this game degenerates into a mere parity argument. If n is even, there is an odd number of moves, so the first player, Geoff, will win. If n is odd, there is an even number of moves, so the second player, Meghan, will win. □