February 2009 Solutions

1) Consider the following polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where a_i is an integer for each $i \in \{0, 1, \ldots, n\}$. If $p(0)$ and $p(1)$ are both odd, show that $p(x)$ has no integer roots, i.e. there does not exist an integer t such that $p(t) = 0$.

Proof. Consider the integer $2N$. Then

$$p(2N) = a_n(2N)^n + a_{n-1}(2N)^{n-1} + \cdots + a_1(2N) + a_0.$$

Then

$$a_n(2N)^n + a_{n-1}(2N)^{n-1} + \cdots + a_1(2N)$$

is an even number, so $p(2N)$ and $p(0) = a_0$ have the same parity (i.e. are both even or are both odd). Since we are given that $p(0)$ is odd, $p(2N)$ is odd for every integer N.

Consider the integer $2N + 1$. Then

$$\begin{aligned} p(2N+1) &= a_n(2N+1)^n + a_{n-1}(2N+1)^{n-1} + \cdots + a_1(2N+1) + a_0 \\ &= T + a_n + a_{n-1} + \cdots + a_1 + a_0 \end{aligned}$$

where T is an even number. Since

$$p(1) = a_n + a_{n-1} + \cdots + a_1 + a_0,$$

we have that $p(1)$ and $p(2N+1)$ have the same parity. Since we are given that $p(1)$ is odd, $p(2N + 1)$ is odd for every integer N.

Thus, no matter what integer t we take, $p(t)$ is always odd. For t to be a root, we must have $p(t) = 0$, but 0 is even. Therefore there are no integer roots of $p(x)$. □

2) Suppose five people each know exactly one piece of information and that each of the five pieces of information are different. Every time person A phones person B, A tells B everything that A knows, while B tells A nothing. Find the minimum number of phone calls between pairs of people required for everyone to know everything. You must show that your answer is a minimum.

Proof. Call the five people p_1, p_2, p_3, p_4 and p_5. Denote a call from p_i to p_j by $p_i \to p_j$.

Consider the following telephone sequence:

$$\begin{aligned} &p_1 \to p_5 \\ &p_2 \to p_5 \\ &p_3 \to p_5 \\ &p_4 \to p_5 \\ &p_5 \to p_4 \\ &p_5 \to p_3 \\ &p_5 \to p_2 \\ &p_5 \to p_1 \end{aligned}$$

Clearly, now all five people know everything. Hence, the minimum number of phone calls is less than or equal to 8.

Now suppose we have a sequence of calls such that at the end, everyone is fully informed. At some point in this sequence, there is a phone call in which the receiver, which we will call P is fully informed, but no one else is. Then each of the other four people must have placed at least one call before this crucial call, otherwise there is no way that P could have received all the information, and each of the other four people must receive at least one more call each after P receives all the information, otherwise, they would never receive all the information. Thus there must be at least 8 telephone calls to disseminate all the information fully.

Since we have shown that there must be at least 8 and that there is a sequence with 8 telephone calls, the minimum number of telephone calls required is 8. □