

January 2009 Solutions

- 1) Suppose a and b are real, positive numbers such that $a + b = 1$. Show the following:

$$\left(1 + \frac{1}{a}\right) \left(1 + \frac{1}{b}\right) \geq 9$$

Proof. Since $a + b = 1$, write $b = 1 - a$. Since a is a positive number, have $a > 0$. Since $a + b = 1$, also have $a < 1$. So, we want to show

$$\left(1 + \frac{1}{a}\right) \left(1 + \frac{1}{1-a}\right) \geq 9$$

Since $0 < a < 1$, have

$$(2a - 1)^2 \geq 0,$$

which implies

$$4a^2 - 4a + 1 \geq 0$$

which implies

$$8a^2 - 8a + 2 \geq 0$$

which implies

$$8a^2 - 8a + 2 + (9a - 9a^2) \geq 9a - 9a^2$$

which implies

$$2 + a - a^2 \geq 9a - 9a^2$$

which implies

$$2 + a - a^2 \geq 9a(1 - a)$$

which implies

$$(a + 1)(1 - a + 1) \geq 9a(1 - a).$$

Since $0 < a < 1$, we have that $a, 1 - a \neq 0$ and $0 < 1 - a < 1$. Thus we can divide both sides of the inequality by $a(1 - a)$ without either dividing by zero or by a negative number. Therefore, we have

$$\begin{aligned} \frac{(a + 1)(1 - a + 1)}{a(1 - a)} &\geq 9 \\ \left(\frac{a + 1}{a}\right) \left(\frac{1 - a + 1}{1 - a}\right) &\geq 9 \\ \left(1 + \frac{1}{a}\right) \left(1 + \frac{1}{1 - a}\right) &\geq 9, \end{aligned}$$

which is what we must show. \square

2) We define the “floor” operation as follows: for any real number t , $\lfloor t \rfloor$ is the greatest integer which is less than or equal to t . For example:

$$\begin{aligned}\lfloor 12 \rfloor &= 12, \\ \lfloor 0.5 \rfloor &= 0, \\ \lfloor \pi \rfloor &= 3, \\ \lfloor -7.643 \rfloor &= -8.\end{aligned}$$

Show that there is no real number x which satisfies the following equation:

$$\lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 8x \rfloor + \lfloor 16x \rfloor + \lfloor 32x \rfloor = 12345.$$

Proof. Let

$$f(t) = \lfloor t \rfloor + \lfloor 2t \rfloor + \lfloor 4t \rfloor + \lfloor 8t \rfloor + \lfloor 16t \rfloor + \lfloor 32t \rfloor.$$

Notice that for $a \leq b$, we have $f(a) \leq f(b)$. That is, $f(t)$ is an increasing function. Also, suppose a is an integer and b is a real number. Then

$$\lfloor a + b \rfloor = a + \lfloor b \rfloor.$$

Suppose $f(x) = 12345$. By inspection, we see that $f(195) = 12285$ and $f(196) = 12348$. Therefore $195 < x < 196$ since $f(t)$ is an increasing function. Let $y = x - 195$. Then

$$\begin{aligned}\lfloor y \rfloor &= \lfloor x - 195 \rfloor = \lfloor x \rfloor - 195 \\ \lfloor 2y \rfloor &= \lfloor 2x - 390 \rfloor = \lfloor 2x \rfloor - 390 \\ &\vdots \\ \lfloor 32y \rfloor &= \lfloor 32x - 6240 \rfloor = \lfloor 32x \rfloor - 6240.\end{aligned}$$

So

$$\begin{aligned}f(y) &= \lfloor y \rfloor + \lfloor 2y \rfloor + \lfloor 4y \rfloor + \lfloor 8y \rfloor + \lfloor 16y \rfloor + \lfloor 32y \rfloor \\ &= (\lfloor x \rfloor - 195) + (\lfloor 2x \rfloor - 390) + \cdots + (\lfloor 32x \rfloor - 6240) \\ &= f(x) - f(195) \\ &= 12345 - 12285 \\ &= 60\end{aligned}$$

But, since $y = x - 195$ and $195 < x < 196$, we have that $0 < y < 1$. Thus

$$\begin{aligned} \lfloor y \rfloor &\leq 0 \\ \lfloor 2y \rfloor &\leq 1 \\ \lfloor 4y \rfloor &\leq 3 \\ \lfloor 8y \rfloor &\leq 7 \\ \lfloor 16y \rfloor &\leq 15 \\ \lfloor 31y \rfloor &\leq 31, \end{aligned}$$

so

$$\begin{aligned} f(y) &= \lfloor y \rfloor + \lfloor 2y \rfloor + \lfloor 4y \rfloor + \lfloor 8y \rfloor + \lfloor 16y \rfloor + \lfloor 31y \rfloor \\ &\leq 0 + 1 + 3 + 7 + 15 + 31 \\ &= 57 \end{aligned}$$

That is, $60 = y \leq 57$, a contradiction. Therefore no such y can exist, so no such x can exist, and so $f(x) = 12345$ has no real solution. \square