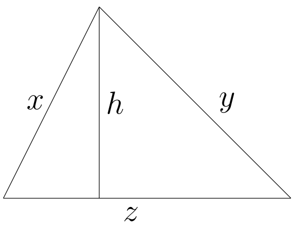
March 2009 Solutions

1) A triangle has the following properties: the three sides and an altitude are four consecutive integers and the altitude partitions the triangle into two right triangles with integer sides. How many triangles have this property?

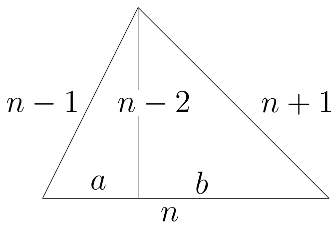
Proof. Let the four consecutive integers be $n-2$, $n-1$, n, and $n+1$.

Draw the triangle below, with sides x, y, and z, and h to denote the altitude.



Since h is the altitude of two right angle triangles, we know that $x > h$ and $y > h$. Since x, y, z, and h are chosen from the set $\{n-2, n-1, n, n+1\}$, this gives us the following four possibilities for sides and altitudes of the triangle.

(a)



Let a and b be as on the diagram. We then have the following three equations:

$$a + b = n$$
$$a^2 + (n-2)^2 = (n-1)^2$$
$$b^2 + (n-2)^2 = (n+1)^2.$$

The last two equations imply

$$(b-a)(b+a) = 4n$$

and since

$$b + a = n,$$

we have

$$\begin{aligned} b - a &= 4 \\ b &= a + 4. \end{aligned}$$

Therefore

$$n = 2a + 4$$

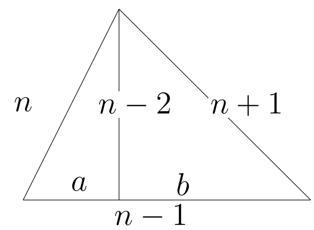
and so

$$\begin{aligned} (2a+2)^2 &= (n-2)^2 \\ &= (n-1)^2 - a^2 \\ &= (2a+3)^2 - a^2 \\ &= 3(a+3)(a+1) \end{aligned}$$

Therefore

$$\begin{aligned} 4(a+1) &= 3(a+3) \\ a &= 5 \\ b &= 9 \\ n &= 14 \\ n-2 &= 12 \\ n-1 &= 13 \\ n+1 &= 15. \end{aligned}$$

(b)

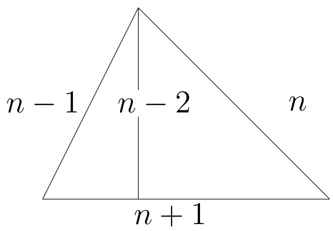


Using similar reasoning to the previous case, we have $b^2 - a^2 = (n+1)^2 - n^2 = 2n + 1$. This gives

$$\begin{aligned} b - a &= \frac{2n+1}{n-1} \\ &= 2 + \frac{3}{n-1} \end{aligned}$$

We know that $n \geq 3$ since $n-2$ and $n-1$ are positive integers. Since $b-a$ is also an integer, we have $n = 4$. thus $b - a = 3$. But $n - 1 = 3$ and $n - 1 = a + b$. Therefore $a = 0$, a contradiction. Therefore this triangle configuration does not work.

(c)



As in the preceding cases, we have

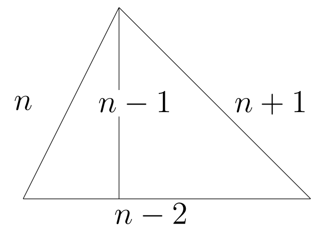
$$b^2 - a^2 = n^2 - (n-1)^2 = 2n - 1$$

which implies

$$\begin{aligned} b - a &= \frac{2n-1}{n+1} \\ &= 2 - \frac{3}{n+1}. \end{aligned}$$

which cannot be an integer since $n \geq 3$ implies $n+1 \geq 4$. Therefore this triangle configuration does not work.

(d)



As in the preceding cases, we have

$$b^2 - a^2 = (n+1)^2 - n^2 = 2n + 1$$

which implies

$$\begin{aligned} b - a &= \frac{2n+1}{n-2} \\ &= 2 + \frac{5}{n-2} \end{aligned}$$

Since $b-a$ is an integer, there are only two possibilities for n, either $n = 3$ or $n = 7$. If $n = 3$, then $b - a = 7$ but $a + b = n - 2 = 1$, a contradiction. If $n = 7$, then $b - a = 3$ and $b + a = 5$, which gives that $a = 1$. Since $n - 1 = 6$, by looking at the left right angle triangle, we get

$$1^2 + 6^2 = 7^2,$$

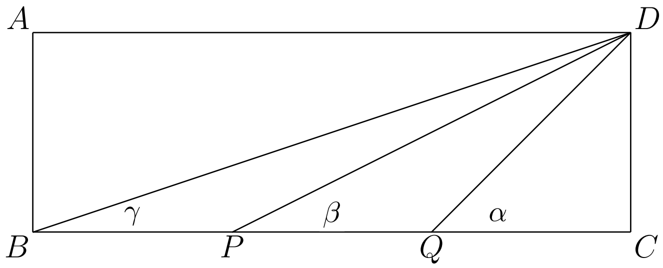
which is untrue. Therefore this triangle configuration does not work.

Thus there is only one such triangle, the one given in the first case. $\square$

2) Let $ABCD$ be a rectangle with $BC = 3AB$. If P and Q are points on BC with $BP = PQ = QC$, show that

$$\angle DBC + \angle DPC = \angle DQC.$$

Proof. Draw our rectangle:



Let α, β, and γ be as defined on the diagram. We are thus trying to show

$$\gamma + \beta = \alpha.$$

We will take $AB = 1$ (since we are dealing with angles, it doesn't matter what side length we chose). We are given

$$AB = BP = PQ = QC = 1.$$

By Pythagorean theorem, we get

$$\begin{aligned} DB &= \sqrt{10} \\ DP &= \sqrt{5} \\ DQ &= \sqrt{2}. \end{aligned}$$

Notice that

$$\begin{aligned} DB &= \sqrt{2} \cdot \sqrt{5} & \qquad PD &= \sqrt{5} \\ BQ &= \sqrt{2} \cdot \sqrt{2} & \qquad DQ &= \sqrt{2} \\ DQ &= \sqrt{2} \cdot 1 & \qquad PQ &= 1 \end{aligned}$$

That is $\triangle DBQ$ and $\triangle PDQ$ are similar triangles and so $\angle PDQ = \angle DBQ = \gamma$. Consider $\triangle PDQ$. The three angles have measurements β, γ, and $180 - \alpha$, and so

$$\begin{aligned} 180 &= \beta + \gamma + (180 - \alpha) \\ \alpha &= \beta + \gamma \end{aligned}$$

as required. $\square$