November 2008 Questions

1) Suppose we are given a list of positive integers with such that

   (i) there is at least one integer in the list;
   (ii) all integers in the list are less than or equal to 2008;
   (iii) repetition of integers is allowed, i.e. an integer can appear more than once in the list;
   (iv) the sum of all the integers in the list is 2008.

   Some such possible lists are $[1, 1, \cdots, 1]$ where the integer 1 appears 2008 times, $[4, 5, 16, 16, 57, 120, 208, 1582]$, and $[2008]$.

   Suppose in our list that there are n_1 integers equal to 1, n_2 integers equal to 2, $\cdots$, n_{2008} integers equal to 2008. What is the largest possible value of

$$n_2 + 2n_3 + 3n_4 + \cdots 2007n_{2008}?$$

   Proof. Note that the number of integers in the list is $n_1 + n_2 + \cdots n_{2008}$. Call this number X_1. Moreover, note that the sum of the integers in the list is $1 \cdot n_1 + 2 \cdot n_2 + \cdots + 2008 \cdot n_{2008}$. Call this number X_2. Then

$$n_2 + 2n_3 + 3n_4 + \cdots 2007n_{2008} = X_2 - X_1$$

   But (iv) gives us that $X_2 = 2003$, so

$$n_2 + 2n_3 + 3n_4 + \cdots 2007n_{2008} = 2008 - X_1$$

   Then the LHS is maximized when X_1 is minimized, and the minimum value of X_1, i.e. the minimum number of integers that can occur in the list, is 1, which, by (iv), must mean that our list is $[2008]$. Then the largest possible value of

$$n_2 + 2n_3 + 3n_4 + \cdots 2007n_{2008}$$

   is 2007. $\square$

2) Let $p(x)$ be a polynomial with the following properties:

   (i) $p(1) = 1$;
   (ii) $\dfrac{p(2x)}{p(x+1)} = 8 - \dfrac{56}{x+7}$ for all values x where both sides are defined.

Show the following:

(a) the degree of $p(x)$ is 3.

(b) $p(-1) = -\dfrac{5}{21}$.

Proof. (a) From (ii):

$$\begin{aligned}\frac{p(2x)}{p(x+1)} &= 8 - \frac{56}{x+7} \\ \frac{(x+7)\cdot p(2x)}{p(x+1)} &= 8(x+7) - 56 \\ \frac{(x+7)\cdot p(2x)}{p(x+1)} &= 8x \\ (x+7)\cdot p(2x) &= 8x \cdot p(x+1)\end{aligned}$$

Since the two sides are equal, both must have the same leading coefficient. If the degree of $p(x)$ is n and the leading coefficient of $p(x)$ is c, then the leading coefficient of $(x+7)\cdot p(2x)$ is $2^n c$ while the leading coefficent of $8x \cdot p(x+1)$ is $8c$. Therefore $2^n c = 8c$, so $2^n = 8$, giving $n = 3$. Therefore the degree of $p(x)$ is 3.

(b) From (a), we have

$$(x+7)\cdot p(2x) = 8x \cdot p(x+1) \tag{1}$$

Note that if $x = 0$, then the RHS of (1) is 0. Thus $7 \cdot p(2 \cdot 0) = 0$, so $p(0) = 0$, meaning that 0 is also a root of $p(x)$. Thus, there exists some polynomial $q(x)$ such that $p(x) = x \cdot q(x)$. Then $p(2x) = 2x \cdot q(2x)$ and so

$$\begin{aligned}(x+7)\cdot p(2x) &= 8x \cdot p(x+1) \\ (x+7)\cdot [2x \cdot q(2x)] &= 8x \cdot p(x+1) \\ 2x(x+7)\cdot q(2x) &= 8x \cdot p(x+1)\end{aligned}$$

Also $p(x+1) = (x+1)q(x+1)$, and substituting that into the above equation gives us

$$\begin{aligned}2x(x+7)\cdot q(2x) &= 8x(x+1)\cdot q(x+1) \\ (x+7)\cdot q(2x) &= 4(x+1)\cdot q(x+1)\end{aligned}$$

Note that $x = -1$ is a root of the RHS of the above equation, so it is also a root of the LHS of the equation. That is $6 \cdot q(-2) = 0$.

Thus -2 is a root of $q(x)$. Thus, there exists some polynomial $r(x)$ such that $q(x) = (x+2)r(x)$. Then $q(2x) = (2x+2)r(2x)$ and $q(x+1) = (x+3)r(x+1)$. Substituting this into our last equation gives us

$$2(x+1)(x+7)r(2x) = 4(x+1)(x+3)r(x+1)$$

and simplifying gives us

$$(x+7)r(2x) = 2(x+3)r(x+1).$$

Note that $x = -3$ is a root of the RHS of the above equation, so it is also a root of the LHS. That is $4r(-6) = 0$, so -6 is a root of $r(x)$. Thus, there exists some polynomial $s(x)$ such that $r(x) = (x+6)s(x)$. Now $p(x) = x \cdot q(x)$ and $q(x) = (x+2)r(x)$ and $r(x) = (x+6)s(x)$, so

$$p(x) = x(x+2)(x+6)s(x).$$

Then the degree of $p(x)$ is at least 3. However, from (a), we know that the degree of $p(x)$ is exactly three, and so $s(x)$ must be a constant, i.e. $s(x) = k$ for some $k \in \mathbb{R}$. That is $p(x) = kx(x+2)(x+6)$. We are given that $p(1) = 1$ and so

$$\begin{aligned}1 &= p(1)\\ 1 &= k \cdot 1 \cdot (1+2) \cdot (1+6)\\ 1 &= k \cdot 21\\ k &= \frac{1}{21}\end{aligned}$$

Therefore

$$p(x) = \frac{x(x+2)(x+6)}{21}$$

and so

$$\begin{aligned}p(-1) &= \frac{(-1)(-1+2)(-1+6)}{21}\\ &= -\frac{5}{21}\end{aligned}$$

as required.

□