Nova Scotia Math League

2010–2011

Game Two

CONTEST PAPER

Team Questions

1) Find a value of b such that the lines $y = 2x + 3$, $y = 4x + 5$, and $y = x + b$ all pass through the same point.

2) How many subsets of the set $\{1, 2, 3, 4, 5\}$ contain either a 1 or a 5 (or both)?

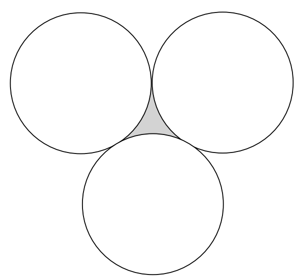
3) A holding tank has an input valve and an output valve. With the input valve open it takes 3 hours to fill the tank from empty. With the output valve open it takes 5 hours to empty a full tank.

   Suppose the tank is empty and the input and output valves are opened simultaneously. How many hours does it take for the tank to become half full?

4) Suppose the function $f(x)$ satisfies $f(1 + 2x) = 3x + 4x^2$ for all x. Find the *sum* of the roots of the equation $f(x) = 0$.

5) Nine balls, numbered 1 through 9, are placed in a bin. If you randomly draw two balls (simultaneously) from the bin, what is the probability that the numbers on those balls are not consecutive?

6) Three circles, each of radius 2, are mutually tangent (see diagram below). Determine the area of the shaded region bounded between the circles.

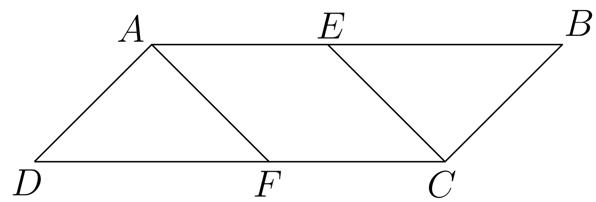


7) Determine the value of the sum

$$i + 2i^2 + 3i^3 + 4i^4 + \cdots + 1001i^{1001},$$

where $i = \sqrt{-1}$, as usual.

8) In the figure below, $ABCD$ and $AECF$ are parallelograms, $\angle DAF = 90°$, and $|AD| = |AE| = |AF|$.



Given that the area of $ABCD$ is 4, find the area of $AECF$.

9) The integers 1 through 2011 (inclusive) are written consecutively to form a new number, as follows:

$$12345678910111213141516 \cdots 200920102011.$$

Find the sum of the digits in this new number.

10) The function g satisfies

$$g(x)g(y) = g(\sqrt{x^2 + y^2}) \qquad \text{for all real numbers } x \text{ and } y.$$

Given that $g(0) = 1$, $g(1) = 2$, and $g(2) = 16$, find the value of $g(3)$.

Pairs Relay

A. John and Andrew being working for Ideas Incorporated on the same day. John agrees to be paid \$20000 for the first year, with his salary increasing by \$10000 every year thereafter. Andrew instead negotiates to be paid \$2000 for the first year, with his salary *doubling* every year thereafter.

Let `A` be such that in the `A`-th year, Andrew's salary exceeds John's for the first time.

Pass on `A`

B. You are given four straws with lengths 2, 3, 6, and `A` inches and are asked to make a triangle from any three of the straws. Let `B` be the number of distinct (i.e. noncongruent) triangles that can be formed.

Pass on `B`

C. Let `C` be the number of pairs of positive integers (n, m) such that

$$mn = 50\texttt{B}.$$

Pass on `C`

D. You will receive `C`.

Alice, Betty, Charlie, and Doug are walking separately along a straight road, never passing one another. At a certain point it turns out that both Charlie and Doug are exactly twice as far from Betty as from Alice, while Alice and Betty are themselves `C` metres apart.

Let `D` be the the distance between Charlie and Doug at this instant.

Done!

Individual Relay

A. Let $\mathtt{A}$ be the coefficient of x^4 in the expansion of $(1+7x-10x^2-7x^3)^2$.

Pass on $\mathtt{A}$

B. You will receive $\mathtt{A}$.

For real numbers x and y, define $x \circ y = \left(\frac{1}{x} + \frac{1}{y}\right)^{-1}$.

Let $\mathtt{B} = \dfrac{1}{((1 \circ \mathtt{A}) \circ (\mathtt{A} \circ 1))}$.

Pass on $\mathtt{B}$

C. You will receive $\mathtt{B}$.

Let $\mathtt{C}$ be such that the line $y = \mathtt{B}x - \mathtt{C}$ intersects the parabola $y = x^2 - x$ exactly once.

Pass on $\mathtt{C}$

D. You will receive $\mathtt{C}$.

Let $\mathtt{D}$ be such that the number "38C69D1" is divisible by 11.

Done!