Nova Scotia Math League

2012–2013

Game One

SOLUTIONS

Team Question Solutions

1. As Angela passes the finish line, Bev has run 8km and Clara has run 6km. So Clara runs $\frac{6}{8} = \frac{3}{4}$ as fast as Clara. In the time it takes Clara has run her last 2km, Clara gets $2 \cdot \frac{3}{4} = \frac{3}{2}$km closer to the finish, leaving her $4 - \frac{3}{2} = \frac{5}{2}$km behind Clara.

2. Suppose there are initially F trees in the forest, and suppose Woody removes T trees. After Woody is done, there are $0.99F - T$ oak trees in the forest, and $F - T$ trees in total. We are told that

$$\frac{0.99F - T}{F - T} = 0.98.$$

   Rearranging gives $0.01F = 0.02T$, so $\frac{T}{F} = \frac{1}{2}$. That is, Woody removed precisely 50% of the trees in the forest.

   Note: The actual number of trees in the forest is immaterial, so there is no loss by simply assuming the forest starts with 100 trees, 99 of these being oak. Then solve $\frac{99-T}{100-T} = 0.98$ to find that Woody cut $T = 50$ of the 100 trees, or 50%..

3. For ease in terminology, let us refer to any valid configuration of numbers inside the grid simply as a *square*. Note that each row and column of a square must contain the numbers 1,2 and 3 in some order. The key to quick counting is to notice that rearranging the rows and/or columns of a square yields another square. In this manner, every square can be transformed into a square in which 1,2,3 appear *in order* along the first row and first column. But there is only one such square, namely

1	2	3
2	3	1
3	1	2

.

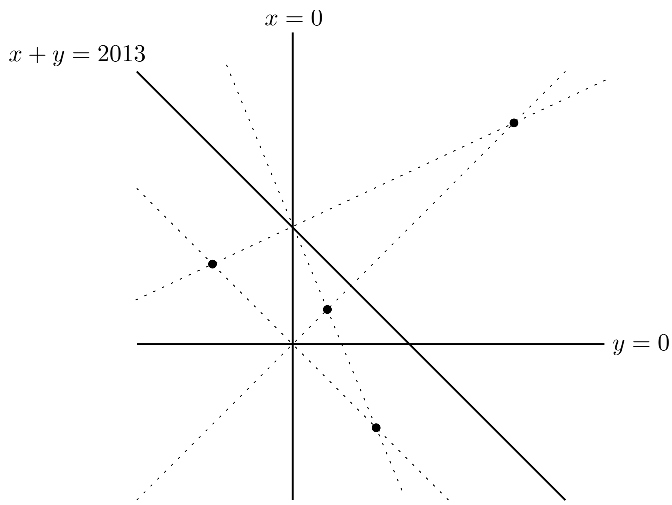
   So to count squares we simply have to find the number of different rearrangements of this particular square! There are $3! = 6$ ways to arrange the rows. Each such arrangement yields a distinct first column, and for any of these arrangements we may further swap the 2nd and 3rd columns. This gives a total of $3! \cdot 2 = 12$ possible squares.

   Note: Configurations of this type of known as *Latin squares*, and have been studied for centuries. Counting Latin squares of arbitrary size is very difficult, but the 3×3 case is small enough that any "organized" method of counting will quickly lead to the correct answer. As with all counting questions, the key is to be organized, so as to avoid misses and double counts.

4. There are $\binom{7}{4} = \frac{7!}{4! \cdot 3!} = 35$ possible arrangements of the balls altogether. To count those in which both end balls are green, simply place two green balls apart on the table, then

arrange 4 red balls and 1 green ball between them. There are clearly 5 ways of doing the latter. Hence the desired probability is $\frac{5}{35} = \frac{1}{7}$.

5. For a point to be equidistant from two intersecting lines, it must lie on one of bisectors of the angles formed between those lines. So to be equidistant from $y = 0$, $x = 0$ and $x + y = 2013$, a point must line on one of the bisectors between $y = 0$ and $x = 0$, and *also* lie on one of the bisectors between $x = 0$ and $x + y = 2013$.[1] The diagram below illustrates the situation, with bisectors shown as dotted lines.



That is, we wish to count the number of intersection points between these two sets of bisectors. Since no pair of bisectors are parallel, each of the two bisectors between $y = 0$ and $x = 0$ will intersect with each of the two bisectors between $x = 0$ and $x + y = 2013$, yielding 4 equidistant points in total. These are marked in the diagram above.

Note: These points are simply the centres of the *incircle* and *excircles* of the triangle formed by the 3 given lines.

6. Since k is odd, we have $f(k) = 3k + 1$, and since $3k + 1$ is even we get $f(f(k)) = \frac{3k+1}{2}$. Since $f(f(f(k))) = 31$, this yields

$$f(\frac{3k+1}{2}) = 31.$$

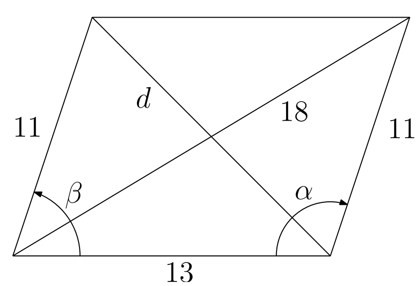
But also note that, by definition, $f(n)$ can only be odd when n is even. Thus the above equality implies $\frac{3k+1}{2}$ is even, so that $f(\frac{3k+1}{2}) = \frac{3k+1}{4}$. Finally, solving $31 = \frac{3k+1}{4}$ yields $k = 41$.

[1]Necessarily such a point will also lie on one of the bisectors between $x + y = 2013$ and $y = 0$.

7. This is a repeated difference of squares factorization, in reverse:

$$
\begin{aligned}
&(\sqrt{2}+\sqrt{3}+\sqrt{5})(-\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}-\sqrt{3}+\sqrt{5})(\sqrt{2}+\sqrt{3}-\sqrt{5})\\
=&\left((\sqrt{3}+\sqrt{5})+\sqrt{2}\right)\left((\sqrt{3}+\sqrt{5})-\sqrt{2}\right)\left(\sqrt{2}-(\sqrt{3}-\sqrt{5})\right)\left(\sqrt{2}+(\sqrt{3}-\sqrt{5})\right)\\
=&\left((\sqrt{3}+\sqrt{5})^2-2\right)\left(2-(\sqrt{3}-\sqrt{5})^2\right)\\
=&\left(2\sqrt{15}+6\right)\left(2\sqrt{15}-6\right)\\
=&2^2\cdot 15-6^2\\
=&24.
\end{aligned}
$$

8. This problem can be solved using Pythagorean theorem and a little bit of algebra, but it's particularly straightforward with the law of cosines. Let d be the length of the unknown diagonal, and let α and β be the interior angles of the parallelogram, as shown in the diagram below.



Then, by the law of cosines, we have

$$18^2 = 11^2 + 13^2 - 2\cdot 11\cdot 13\cdot\cos\alpha$$

and

$$d^2 = 11^2 + 13^2 - 2\cdot 11\cdot 13\cdot\cos\beta.$$

But $\beta = 180^\circ - \alpha$, so $\cos\beta = -\cos\alpha$. Therefore, summing the above two equations yields

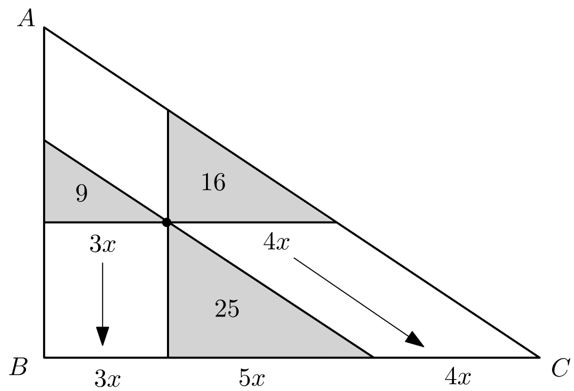
$$d^2 + 18^2 = 2(11^2 + 13^2).$$

Evaluating now gives $d^2 = 256$, hence $d = 16$.

Note: The cosine law isn't magical. To prove it you simply drop a perpendicular, apply the Pythagorean theorem, and do a little algebra — exactly as you could do to solve this question without the cosine law!

9. By construction, each of the small shaded triangles is similar to the large triangle. Since the areas of the three small triangles are in the ratios $9 : 16 : 25$, their corresponding

sides must be in the ratios $3:4:5$. Let their bases have lengths $3x$, $4x$, and $5x$, for some value of x, and "slide" these lengths down as indicated in the diagram below to see that the base of the big triangle has length $3x+4x+5x=12x$.



It follows that the area of the big triangle is $12^2=144$.

Note: Notice that the fact that $B=90^\circ$ and $|AB|:|BC|=3:4$ is irrelevant in this problem. The key here is simply that for similar figures, area grows quadratically with perimeter. That is, if F_1 and F_2 are two similar figures, and if the linear dimension (base / height / perimeter) of F_1 is α times that of F_2, then the area of F_1 is α^2 times that of F_2.

10. Let the intersection points of the line with the ellipse be (x_1,y_1) and (x_2,y_2). Then the point we want is $(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2})$.

Substitute $y=x+1$ into the equation of the ellipse to get

$$\frac{x^2}{2012}+\frac{(x+1)^2}{2012}=1,$$

and note that the roots of this quadratic are precisely x_1 and x_2, so the *sum* of its roots is x_1+x_2. Expanding gives

$$\left((\frac{1}{2012}+\frac{1}{2013}\right)x^2+\frac{2}{2013}x+\left(\frac{1}{2002}-1\right)=0,$$

and therefore (using the fact that the sum of the roots of the quadratic $ax^2+bx+c=0$ is simply $-\frac{b}{a}$) we obtain

$$\frac{x_1+x_2}{2}=\frac{\frac{-2}{2013}}{2\left(\frac{1}{2012}+\frac{1}{2013}\right)}=-\frac{2012}{4025}.$$

This is the x-coordinate of the desired midpoint. Since the midpoint lies on the line $y=x+1$, its y-coordinate is simply $-\frac{2012}{4024}+1=\frac{2013}{4025}$.

Pairs Relay Solutions

P-A. Suppose M has tens digit a and ones digit b, so that $M = 10a + b$. Then $M = S(M) + P(M)$ is equivalent to $10a + b = a + b + ab$, which leads to $9a = ab$, or simply $b = 9$ (since a cannot be zero). Thus $\mathtt{A} = 9$.

P-B. Alice and Bob can each make any of $\mathtt{A}$ choices, so there are $\mathtt{A}^2$ possible outcomes. Of these, there are exactly $\mathtt{A}$ in which both Bob and Alice make the same choice (i.e. they could each choose 1, or each choose 2, etc.). Thus the desired probability is $\mathtt{B} = 1 - \frac{\mathtt{A}}{\mathtt{A}^2} = 1 - \frac{1}{\mathtt{A}}$. With $\mathtt{A} = 9$ we have $\mathtt{B} = \frac{8}{9}$.

P-C. We have

$$\mathtt{C} = \frac{x}{y} + \frac{y}{x} = \frac{x^2 + y^2}{xy} = \frac{(x+y)^2 - 2xy}{xy} = \frac{2^2 - 2\mathtt{B}}{\mathtt{B}} = \frac{4}{\mathtt{B}} - 2.$$

With $\mathtt{B} = \frac{8}{9}$ this gives $\mathtt{C} = \frac{5}{2}$.

P-D. The answer will be n plus the number of *distinct* squares and cubes less than or equal to n, provided there are no squares or cubes between n and this number!

In our case, $n = 50 \cdot \frac{5}{2} = 125$, so there are 11 squares $(1, 4, \ldots, 100, 121)$ and 5 cubes $(1, 8, 27, 64, 125)$ less than or equal to n. However, the cubes 1 and 64 also appear in the squares list, so there are only $11 + 3 = 14$ distinct squares and cubes in the appropriate range. Since there are no further squares or cubes between $n = 125$ and , $n + 14 = 139$, we conclude that $= 139$.

Individual Relay Solutions

I-A. There are $\mathtt{A} = 12$ such arrangements, which can be counted by hand or by various different arguments. Here's one: Imagine four blank spaces in a row. Pick two of them that are not side-by-side in any of three ways. Place your vowel in these spaces in either or two orders (A-O or O-A). Place the consonants in the remaining spaces in either of two orders (B-T or T-B). This gives a total of $3 \cdot 2 \cdot 2 = 12$ arrangements.

I-B. Multiply the given equation by $\mathtt{AB}$ and rearrange to get

$$2\mathtt{B}^2 + \mathtt{AB} - \mathtt{A}^2 = 0.$$

Solve using the quadratic formula to get

$$\mathtt{B} = \frac{-\mathtt{A} \pm \sqrt{9\mathtt{A}^2}}{4} = \frac{-AA \pm 3\mathtt{A}}{4}.$$

Thus $\mathtt{B} = \frac{\mathtt{A}}{2}$ or $\mathtt{B} = -\mathtt{A}$, whichever one is positive. Since $\mathtt{A} = 12$, we get $\mathtt{B} = \frac{12}{2} = 6$.

I-C. The lines $y = -3x$ and $y = 6x$ intersect $y = \mathtt{B}$ at $x = -\frac{\mathtt{B}}{3}$ and $x = \frac{\mathtt{B}}{6}$, respectively. Thus the triangle in question has base $|\frac{\mathtt{B}}{6} - (-\frac{\mathtt{B}}{3})| = \frac{|\mathtt{B}|}{2}$ and height $|\mathtt{B}|$, so it has area

$$\mathtt{C} = \frac{1}{2} \cdot \frac{|\mathtt{B}|}{2} \cdot |\mathtt{B}| = \frac{\mathtt{B}^2}{4}.$$

With $\mathtt{B} = 6$ we get $\mathtt{C} = 9$.

I-D. Notice that

$$\begin{aligned} xyz + xy + xz + yz &= (x+1)(y+1)(z+1) - (x+y+z) - 1 \\ &= (x+1)(y+1)(x+1) - \mathtt{C} - 1, \end{aligned}$$

since we are given $x + y + z = \mathtt{C}$. So to maximize this product we wish to maximize $(x+1)(y+1)(z+1)$ subject to $x + y + z = \mathtt{C}$. This will be accomplished when $x + 1, y + 1$ and $z + 1$ are as close to each other as possible. Since $\mathtt{C} = 9$, we find that we can in fact let all three quantities be *equal* by taking $x = y = z = 3$. This gives a maximum value of $4^3 - 9 - 1 = 54$.

Note: There is really no need to factor here. The symmetry of the given expression suggests that it should reach its optimum when $x = y = z$. The only catch would be if $\mathtt{C}$ were not divisible by 3. What then?