Nova Scotia
Math League

2017–2018

Game Three

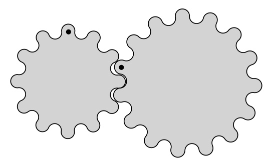
Problems and Solutions

Team Questions

1. Find the percentage increase in the quantity x^2/y^2 when x is increased by 8% and y is decreased by 10%.

   Solution: Since $(\frac{108}{90})^2 = (\frac{6}{5})^2 = \frac{36}{25} = \frac{144}{100}$, it increases by 44%. □

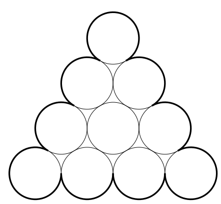
2. The smaller gear below begins to turn at $\frac{1}{4}$ revolution per minute. How many minutes elapse before both gears return to their starting positions?



   Solution: The gears have 12 and 15 teeth, respectively. So they reach their starting positions after $\operatorname{lcm}(12, 15) = 60$ teeth have engaged. This takes $4 \cdot \frac{60}{12} = 20$ minutes.

   □

3. Each of the circles in the figure below has circumference 1. Find the perimeter of the figure (highlighted in the diagram).



   Solution: The perimeter consists of two semicircles per side of the triangle, plus $\frac{5}{6}$ of a circle at each corner, so $3(2 \cdot \frac{1}{2}) + 3(\frac{5}{6}) = \frac{11}{2}$.

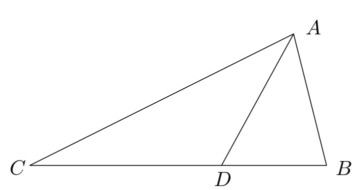
   □

4. It takes 5 minutes to fill my bathtub and 7 minutes for it to drain empty. How many minutes will it take to fill the bathtub if I leave the drain open while filling it?

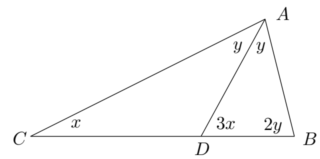
   Solution: The fill and drain rates are $1/5$ and $1/7$ tubs per minute, respectively. With the drain open, the tub fills at a rate of $\frac{1}{5} - \frac{1}{7} = \frac{2}{35}$ tubs per minute, whose reciprocal gives $\frac{35}{2} = 17.5$ minutes per tub.

   □

5. Triangle $\triangle ABC$ is isosceles, with $|AC| = |BC|$. The bisector of $\angle CAB$ meets BC at D, and $\angle ADB = 3\angle ACB$. Find the degree measure of $\angle ACB$.



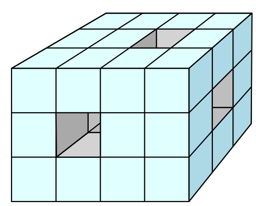
Solution: Let x and y be the degree measures of $\angle BAD$ and $\angle ACB$, respectively, giving the configuration shown below. Summing the angles in triangles $\triangle ABD$ and $\triangle ABC$ yields $3x + 3y = 180$ and $x + 4y = 180$, respectively. Solve to obtain $x = 20$.



Alternatively: Let x be the degree measure of $\angle ACB$, so that $\angle ADB = 3x$. The exterior angle theorem implies $\angle DAC + \angle ACB = \angle ADB$, whence $\angle DAC = 3x - x = 2x$. Then $\angle DAB = 2x$ since AD bisects $\angle CAB$, and $\angle ABC = \angle BAC = 2x + 2x = 4x$ since $\triangle ABC$ is isosceles. Summing the angles in $\triangle ABC$ gives $9x = 180$, so $x = 20°$

□

6. A $3 \times 3 \times 4$ block is created by gluing together several unit cubes. Three 1×1 square tunnels are then bored completely through the cube as shown below. (The tunnels are perpendicular to the faces.) Find the surface area of this solid.



Solution: The original surface area is $2(3^2) + 4(4 \cdot 3) = 66$. There are three tunnels, of lengths 3, 3, and 4. If these didn't intersect, their net effect on surface area would be $(3 + 3 + 4)(4) - 3(2) = 34$. But the tunnels intersect pairwise, with each intersection decreasing the surface area by 4, for a net effect of $-3(4)$. Thus the surface area is $66 + 34 - 12 = 88$.

Note: There are many ways to organize one's counting. One clever method entails looking "through" the solid along each of the 3 axes, as with x-ray vision, and noting how many faces are encountered.

□

7. Eight friends want to split into 4 pairs to play a game. In how many ways can this be done?

Solution: There are $7 \cdot 5 \cdot 3 \cdot 1 = 105$ possibilities: Pair the oldest boy with any of the 7 remaining, then pair the oldest of the remaining boys with any of the other 5, etc. (The point of choosing the "oldest" at each stage is to assert that we have a definitive choice in mind at each stage, which ensures that we aren't over-counting.)

Alternative solution: Label the boys 1 through 8. Permute them in any of 8! ways, and create pairs by taking the first two, the second two, etc. Now divide by $(2!)^4$ to eliminate the ordering within each pair, and again by 4! to eliminate ordering of the pairs. The result is $\frac{8!}{(2!)^4\,4!} = 105$.

□

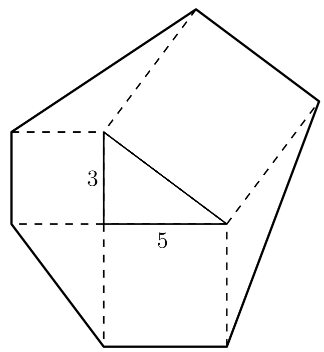
8. For how many integers n between 1 and 100 (inclusive) is $2018^n - 2017^n$ divisible by 5?

Solution: The units digits of 2018^n and 2017^n, for $n = 1, 2, 3, \ldots$ form periodic sequences $8, 4, 2, 6, 8, 4 \ldots$ and $7, 9, 3, 1, 7, 9, \ldots$, respectively. Thus the units digit of $2018^n - 2017^n$ follows the period sequence $1, 5, 9, 5, 1, 5, \ldots$. This difference is divisible by 5 if and only if n is even. There are $\frac{100}{2} = 50$ such values of n between 1 and 100.

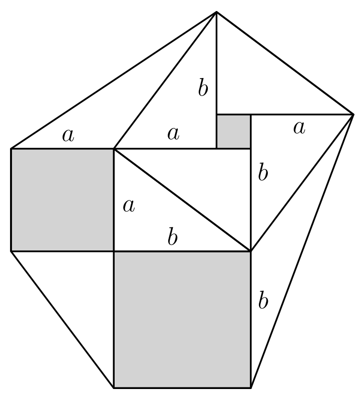
Alternative solution: Performing arithmetic modulo 5, we have $2018^n - 2017^n \equiv (-2)^n - 2^n \equiv 2^n((-1)^n - 1)$. This is clearly 0 for even n and nonzero for odd n (since 5 is prime). So again there are 50 values of n.

□

9. Squares are extended from the sides of a right triangle with legs of lengths 3 and 5. The vertices of the square are then adjoined to form an irregular hexagon, as shown. Find the area of the hexagon.

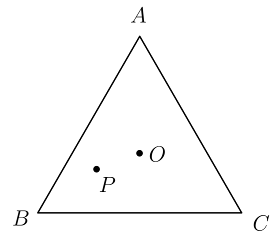


Solution: Say the legs of the triangle are a and b. Divide the hexagon as shown below into 8 triangles, each of area $\frac{1}{2}ab$, and three squares (shaded)s with areas a^2, b^2, and $(a-b)^2$. Thus the total area is $4ab + a^2 + b^2 + (a-b)^2 = 2(a^2 + ab + b^2)$. s With $a = 3$ and $b = 5$ get area $2(9 + 15 + 25) = 98$.

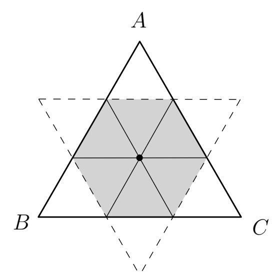


□

10. Let O be the centre of equilateral triangle $\triangle ABC$ (i.e. the unique point equidistant from each vertex). Another point P is selected at random in the interior of $\triangle ABC$. Find the probability that P is closer to O than it is to any of A, B or C.



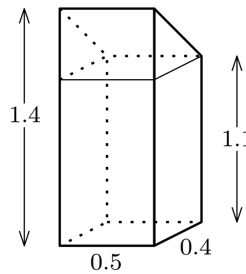
Solution: Draw perpendicular bisectors of AO, OB and OC. Then P is closest to O provided it is in the intersection of these half-planes, which is a hexagon inside ABC. Subdivide ABC into 9 equilateral triangles to see that the area of the hexagon is $2/3$ that of the whole.



□

Pairs Relay

P-A. A simplified schematic of a mailbox is shown below, with all measurements in metres.



Let A be the volume of the mailbox, in cubic metres. **Pass on A**

Solution: The facing trapezoid has area $\frac{1}{2}(1.1 + 1.4)(0.4) = \frac{5}{4} \cdot \frac{2}{5} = \frac{1}{2}$. So the volume is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. □

P-B. You will receive A. Let $n = 40\mathsf{A}$. (This should be an integer.)

Let B be the units digit of 123^n. **Pass on B**

Solution: The units digits of 3^n, for $n = 0, 1, 2, 3, \ldots$, form a periodic sequence $1, 3, 9, 7, 1, 3, 9, 7, \ldots$. Since $n = 40(\frac{1}{4}) = 10$ leaves remainder 2 upon division by 4, the units digit of 3^n is $\mathsf{B} = 9$.

□

P-C. You will receive B.

Suppose $x + y = \mathsf{B}$, $x + z = 2\mathsf{B}$ and $y + z = 3\mathsf{B}$.

Let $\mathsf{C} = x + y + z$. **Pass on C**

Solution: Sum to get $x + y + z = \frac{1}{2}(\mathsf{B} + 2\mathsf{B} + 3\mathsf{B}) = 3\mathsf{B}$. With $\mathsf{B} = 9$ get $\mathsf{C} = 27$. □

P-D. You will receive C.

Jessica plans to give each of her friends a bag of 30 candy hearts for Valentine's Day. She labels some bags (one for each friend) and begins filling them one at a time. Unfortunately, she finds that she only has C hearts remaining for the last bag. So she instead decides to give each friend only 29 hearts and she keeps the 10 leftovers for herself.

Let D be the number of hearts Jessica began with. **Done!**

Solution: Suppose Jessica has x friends. Then $\mathsf{D} = 30(x - 1) + \mathsf{C} = 29x + 10$. Solve to get $x = 40 - \mathsf{C}$, so $\mathsf{D} = (30)(39 - \mathsf{C}) + \mathsf{C}$. With $\mathsf{C} = 27$ get $\mathsf{D} = 387$.

□

Individual Relay

I-A. Colin, John, and Marc went camping. Over the course of the trip, Marc paid \$93 for food, Colin paid \$58 for gas, and John paid \$53 for the campsite. Both John and Colin gave Marc some money so as to split the costs of the trip equally.

Let A be the amount (in dollars) John gave Marc. **Pass on A**

Solution: The total cost was $93 + 58 + 53 = 204$, so the per-person cost is $\frac{204}{3} = 68$. John paid \$53 already so he owes Marc $\mathsf{A} = 68 - 53 = 15$.

□

I-B. You will receive A.

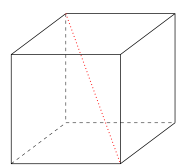
A 250 metre long train travels at a constant speed of 90 km/h. The train enters a tunnel and fully emerges A seconds later.

Let B be the length of the tunnel, in metres. **Pass on B**

Solution: We have $(\mathsf{B} + 250)/\mathsf{A} = 90 \cdot \frac{1000}{3600} = 25$. Thus $\mathsf{B} = 25\mathsf{A} - 250$. With $\mathsf{A} = 15$ get $\mathsf{B} = 125$. □

I-C. You will receive B.

A *main diagonal* of a cube connects two opposing vertices, as shown:



Suppose the volume of a cube is B cm^3. Let C be the length (in cm) of its main diagonal, **rounded to the nearest integer**.

Pass on C

Solution: Let $x = \sqrt[3]{\mathsf{B}}$. Then the diagonal has length $\sqrt{x^2 + x^2 + x^2} = x\sqrt{3}$. If $\mathsf{B} = 125$ then $x = 5$ and thus $x\sqrt{3} \approx 5 \cdot \frac{7}{4} = \frac{35}{4} = 8.75$. Thus $\mathsf{C} = 9$.

□

I-D. You will receive C.

Suppose $x : y = 2 : 1$ and $y : z = \mathsf{C} : 2$.

Let $\mathsf{D} = \dfrac{y}{x + z}$. **Done!**

Solution: Get $\mathsf{D} = 1/(\frac{x}{y} + \frac{z}{y}) = 1/(2 + \frac{2}{\mathsf{C}}) = \frac{\mathsf{C}}{2\mathsf{C}+2}$. With $\mathsf{C} = 9$ get $\mathsf{D} = \frac{9}{20}$. □